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"First–order second–class systems"

–Summary of Ph.D. thesis–

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1 Main problem approached in the thesis

Let $z^A\big|_{A=1,\ldots,2n}$ be a set of bosonic variables that parameterize the time-evolution of a dynamical system. Assume the considered system is described by a first-order action

$$S_0\left[z^A\right] = \int_{t_1}^{t_2} dt\left(a_A\left(z\right)\dot{z}^A - V\left(z\right)\right) \equiv \int_{t_1}^{t_2} dt L_0\left(z, \dot{z}\right), \tag{1}$$

with $a_A\left(z\right)$ the one-form potential and $V\left(z\right)$ a given potential. In what follows we consider the case where the symplectic two-form

$$\omega_{AB}\left(z\right) = \frac{\partial a_B}{\partial z^A} - \frac{\partial a_A}{\partial z^B} = -\omega_{BA}\left(z\right) \tag{2}$$

is non-degenerate, i.e., $\det\omega_{AB} \neq 0$ (locally). The non-degeneracy of the symplectic two-form leads to the existence of a bracket structure, locally given by

$$\left[F_1, F_2\right] = \omega^{AB}\frac{\partial F_1}{\partial z^A}\frac{\partial F_2}{\partial z^B}, \tag{3}$$

in terms of which the Euler–Lagrange equations of motion [1], $\frac{\delta L_0}{\delta z^A} \equiv \frac{\partial L_0}{\partial z^A} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{z}^A}\right) = 0$, deriving from the first-order variational principle based on (1), can be put in the form

$$H^A \equiv \dot{z}^A - \left[z^A, V\right] = 0, \tag{4}$$

with ω^{AB} the inverse of ω_{AB}. The fixation of integration constants in the general solution of equations (4) requires to impose the initial conditions

$$z^A\left(t_0\right) = z_0^A,\ t_1 \leq t_0 \leq t_2. \tag{5}$$

On the one hand, it is easy to see that the Lagrangian from (1) is degenerate in the sense of the Dirac approach [2]–[4], but the canonical analysis of this Lagrangian emphasizes only second-class constraints. Then, by passing to the Dirac bracket we find that the dynamics in terms of independent variables is precisely described by equations (4). In the context of first-order systems, an alternative viewpoint to constrained systems has been formulated in [5]. On the other hand, equations (4) may be regarded as the Hamilton equations [6] for a system with the Hamiltonian $H_0\left(z\right) \equiv V\left(z\right)$. Thus, given a first-order formulation of dynamics, we always find that the Euler–Lagrange and Hamilton equations expressed in terms of the same variables coincide. Actually, we have that $\frac{\delta L_0}{\delta z^A} = \omega_{AB} H^B$.

The main problem approached in the thesis is the following: given a first-order formulation of dynamics in terms of some variables, does there exist an equivalent, non-degenerate, second-order Lagrangian formulation in terms of the same variables?

2 Results

In the sequel we will briefly present the main results of the thesis.

The first main result is given by Theorems 1–2.

Theorem 1 *For any first-order Lagrangian $L_0(z, \dot{z}) = a_A(z)\dot{z}^A - V(z)$ with a non-degenerate symplectic two-form there exists a second-order Lagrangian $\bar{L}_0(z, \dot{z}) = \frac{1}{2}k_{AB}\dot{z}^A\dot{z}^B - \bar{V}(z)$ such that*

$$\left\{ \begin{array}{c} \frac{\delta L_0}{\delta z^A} = 0, \\ z^A(t_0) = z_0^A, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} \frac{\delta \bar{L}_0}{\delta z^A} = 0, \\ z^A(t_0) = z_0^A,\ \dot{z}^A(t_0) = \left[z^A, V\right]\big|_{z_0^A}, \end{array} \right. \tag{6}$$

if and only if there exists a constant, symmetric, and invertible matrix k_{AB} such that the relations

$$k_{AC}\frac{\partial\left[\left[z^C, V\right], V\right]}{\partial z^B} = k_{BC}\frac{\partial\left[\left[z^C, V\right], V\right]}{\partial z^A}. \tag{7}$$

are fulfilled.

Theorem 1 proves the existence of a Lagrangian formulation for the dynamics of a first-order second-order system with the following properties: a) it is non-degenerate and second-order; b) it is equivalent to the first-order formulation (in terms of the same variables) at the level of the solutions to the equations of motion subject to some mutually compatible initial conditions.

From the above theorem it follows that $\bar{V}(z)$ is solution to the equations

$$-\frac{\partial\bar{V}}{\partial z^A} = k_{AC}\left[\left[z^C, V\right], V\right], \tag{8}$$

such that

$$\frac{\delta\bar{L}_0}{\delta z^A} = -k_{AB}E^B \equiv -k_{AB}\left(\ddot{z}^B - \left[\left[z^B, V\right], V\right]\right). \tag{9}$$

In the sequel we pass to field theories, described by first-order actions of the type

$$S_0\left[Q^A\right]=\int d^Dx\left(\alpha_A\left(Q\right)\dot{Q}^A-\mathcal{V}\left(Q^A,\partial_iQ^A\right)\right)\equiv\int d^Dx\mathcal{L}_0\left(Q^A,\partial_\mu Q^A\right). \tag{10}$$

In (10) we have used the standard notations $\dot{f}=\partial_0 f=\partial f/\partial t$ and $\partial_i g=\partial g/\partial x^i$, and the flat Minkowski metric of 'mostly minus' signature, $\sigma_{\mu\nu}=\sigma^{\mu\nu}=(+-\ldots-)$. We consider again that the symplectic two-form

$$\Omega_{AB}\left(x\right)=\frac{\partial\alpha_B}{\partial Q^A}\left(x\right)-\frac{\partial\alpha_A}{\partial Q^B}\left(x\right)=-\Omega_{BA}\left(x\right) \tag{11}$$

is non-degenerate, which leads to the bracket structure

$$\left[F\left(x^0\right),G\left(x^0\right)\right]=\int d^{D-1}z\frac{\delta F}{\delta Q^A\left(x^0,\mathbf{z}\right)}\Omega^{AB}\left(x^0,\mathbf{z}\right)\frac{\delta G}{\delta Q^B\left(x^0,\mathbf{z}\right)}, \tag{12}$$

with Ω^{AB} the inverse of Ω_{AB}. The field equations deriving from (10) read as

$$\mathcal{H}^A\equiv\dot{Q}^A\left(x\right)-\left[Q^A\left(x\right),V\left(x^0\right)\right]=0, \tag{13}$$

with $V\left(x^0\right)=\int d^{D-1}x\mathcal{V}\left(Q^A,\partial_iQ^A\right)$. Regarding equations (13) we take the initial conditions

$$Q^A\left(t_0,\mathbf{x}\right)=Q_0^A\left(\mathbf{x}\right). \tag{14}$$

Along the same line with the finite-dimensional case, we arrive at the following theorem.

Theorem 2 *For any first-order Lagrangian $\mathcal{L}_0\left(Q^A,\partial_\mu Q^A\right)=\alpha_A\left(Q\right)\dot{Q}^A-\mathcal{V}\left(Q^A,\partial_iQ^A\right)$ with a non-degenerate symplectic two-form there exists a second-order Lagrangian $\bar{\mathcal{L}}_0\left(Q^A,\partial_\mu Q^A\right)=\frac{1}{2}\rho_{AB}\dot{Q}^A\dot{Q}^B-\bar{\mathcal{V}}\left(Q^A,\partial_iQ^A\right)$ such that*

$$\left\{\begin{array}{c}\frac{\delta\mathcal{L}_0}{\delta Q^A}=0,\\ Q^A\left(t_0,\mathbf{x}\right)=Q_0^A\left(\mathbf{x}\right),\end{array}\right.\Leftrightarrow\left\{\begin{array}{c}\frac{\delta\bar{\mathcal{L}}_0}{\delta Q^A}=0,\\ Q^A\left(t_0,\mathbf{x}\right)=Q_0^A\left(\mathbf{x}\right),\\ \dot{Q}^A\left(t_0,\mathbf{x}\right)=\left.\left[Q^A\left(x\right),V\left(x^0\right)\right]\right|_{Q_0^A\left(\mathbf{x}\right)},\end{array}\right. \tag{15}$$

if and only if there exists a constant, symmetric, and invertible matrix ρ_{AB} such that the relations

$$\rho_{AC}\frac{\delta\left[\left[Q^C,V\right],V\right]}{\delta Q^B}\left(x\right)=\rho_{BC}\frac{\delta\left[\left[Q^C,V\right],V\right]}{\delta Q^A}\left(x\right) \tag{16}$$

are fulfilled.

Theorem 2 extends the result of Theorem 1 to field theories. Moreover, we can show that $\bar{\mathcal{V}}\left(Q^{A},\partial_{i}Q^{A}\right)$ is solution to the equations

$$\frac{\delta\bar{\mathcal{V}}}{\delta Q^{A}}\left(x\right)=-\rho_{AC}\left[\left[Q^{C}\left(x\right),V\left(x^{0}\right)\right],V\left(x^{0}\right)\right], \tag{17}$$

from which we obtain that

$$\frac{\delta\bar{\mathcal{L}}_{0}}{\delta Q^{A}}=-\rho_{AB}\mathcal{E}^{B}\equiv-\rho_{AB}\left(\ddot{Q}^{B}\left(x\right)-\left[\left[Q^{B}\left(x\right),V\left(x^{0}\right)\right],V\left(x^{0}\right)\right]\right). \tag{18}$$

Our results can be easily managed to cover the case of fermionic degrees of freedom by introducing some phase factors and right or left derivatives.

Examples

e_{1}) Let us take the case of scalar field theories described by the first-order Lagrangian

$$\mathcal{L}_{0}=\dot{\varphi}^{a}\pi_{a}-\frac{1}{2}\mu^{ab}\pi_{a}\pi_{b}+\frac{1}{2}\mu_{ab}\left(\partial_{i}\varphi^{a}\right)\left(\partial^{i}\varphi^{b}\right)-Z\left(\varphi^{a}\right), \tag{19}$$

where μ^{ab} is a constant, symmetric, and invertible matrix, μ_{ab} is the inverse of μ^{ab}, and $Z\left(\varphi^{a}\right)$ is an arbitrary function depending only on the undifferentiated scalar fields. The corresponding second-order Lagrangian reads as

$$\bar{\mathcal{L}}_{0}=\tilde{c}\left(\left(\partial_{\mu}\varphi^{a}\right)\left(\partial^{\mu}\pi_{a}\right)-\mu^{ab}\pi_{a}\frac{\partial Z}{\partial\varphi^{b}}\right), \tag{20}$$

where $\tilde{c}$ is a real constant.

e_{2}) For the Schrödinger Lagrangian with a time-independent potential

$$\mathcal{L}_{0}=i\hbar\psi^{*}\dot{\psi}+\psi^{*}\left(\frac{\hbar^{2}}{2m}\partial_{i}\partial_{i}-U\left(\mathbf{x}\right)\right)\psi, \tag{21}$$

we find that

$$\bar{\mathcal{L}}_{0}=\alpha\left(\hbar\dot{\psi}^{*}\dot{\psi}-\frac{1}{\hbar}\psi^{*}\left(\frac{\hbar^{2}}{2m}\partial_{i}\partial_{i}-U\left(\mathbf{x}\right)\right)^{2}\psi\right), \tag{22}$$

with α a real constant.

e_{3}) The Dirac Lagrangian

$$\mathcal{L}_{0}=\bar{\psi}_{a}\left(\mathrm{i}\left(\gamma^{\mu}\right)^{a}{}_{b}\left(\partial_{\mu}\psi^{b}\right)-m\psi^{a}\right), \tag{23}$$

leads to the Klein–Gordon Lagrangian

$$\bar{\mathcal{L}}_0 = \beta \left(\left(\partial_\mu \bar{\psi}_a \right) \partial^\mu \psi^a - m^2 \bar{\psi}_a \psi^a \right), \tag{24}$$

with β an arbitrary, non-vanishing real constant.

e_4) Finally, the Lagrangian of gauge-fixed massless vector fields

$$\mathcal{L}_0 = \dot{A}^\mu \pi_\mu + \frac{1}{2} \pi_\mu \pi^\mu - \frac{1}{2} \left(\partial_i A_\nu \right) \partial^i A^\nu, \tag{25}$$

yields the second-order Lagrangian

$$\bar{\mathcal{L}}_0 = \gamma \left(\partial_\mu A_\nu \right) \partial^\mu \pi^\nu, \tag{26}$$

with γ an arbitrary, non-vanishing real constant.

The second main result is synthesized by Theorems 3–4.

Theorem 3 *For any first-order Lagrangian $L_0 (z, \dot{z}) = a_A (z) \dot{z}^A - V (z)$ with a nondegenerate symplectic two-form there exists a second-order Lagrangian $\hat{L}_0 (z, \dot{z}) = \frac{1}{2} k_{AB} \left(\dot{z}^A - \left[z^A, V \right] \right) \left(\dot{z}^B - \left[z^B, V \right] \right)$ such that*

$$\left\{ \begin{array}{c} \frac{\delta L_0}{\delta z^A} = 0, \\ z^A (t_0) = z^A{}_0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} \frac{\delta \hat{L}_0}{\delta z^A} = 0, \\ z^A (t_0) = z^A{}_0, \ \dot{z}^A (t_0) = \left[z^A, V \right] \big|_{z^A{}_0} \end{array} \right. , \tag{27}$$

where k_{AB} is a constant, symmetric, and invertible matrix.

It is easy to see that the Euler–Lagrange equations that derive from $\hat{L}_0$ are expressed by

$$\begin{aligned} \hat{E}_A \equiv & \frac{\delta \hat{L}_0}{\delta z^A} \equiv -k_{AB} \ddot{z}^B + \left(k_{AC} \frac{\partial \left[z^C, V \right]}{\partial z^B} - k_{BC} \frac{\partial \left[z^C, V \right]}{\partial z^A} \right) \dot{z}^B + \\ & + k_{BC} \left[z^B, V \right] \frac{\partial \left[z^C, V \right]}{\partial z^A} = 0. \end{aligned} \tag{28}$$

Theorem 3 proves the existence of a Lagrangian formulation for the dynamics of a first-order second-class system with the following properties: a) it is non-degenerate and second-order; b) it is equivalent to the first-order formulation (in terms of the same variables) at the level of the solutions to the equations of motion subject to some mutually compatible initial conditions. We remark that the previous theorem holds under some more relaxed conditions than those of Theorem 1 (matrix k_{AB} is no longer restricted to satisfy conditions (7)).

Along the same line like in the finite-dimensional case, we arrive at the following theorem.

Theorem 4 *For any first-order Lagrangian* $\mathcal{L}_0\left(Q^A,\partial_\mu Q^A\right)=\alpha_A(Q)\dot{Q}^A-\mathcal{V}\left(Q^A,\partial_i Q^A\right)$ *with non-degenerate symplectic two-form, there exists a second-order Lagrangian* $\hat{\mathcal{L}}_0\left(Q^A,\partial_\mu Q^A\right)=\frac{1}{2}\rho_{AB}\left(\dot{Q}^A-\left[Q^A,V\right]\right)\left(\dot{Q}^B-\left[Q^B,V\right]\right)$ *such that*

$$\left\{\begin{array}{c}\frac{\delta\mathcal{L}_0}{\delta Q^A}=0,\\ Q^A(t_0,\mathbf{x})=Q^A{}_0(\mathbf{x}),\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}\frac{\delta\hat{\mathcal{L}}_0}{\delta Q^A}=0,\\ Q^A(t_0,\mathbf{x})=Q^A{}_0(\mathbf{x}),\\ \dot{Q}^A(t_0,\mathbf{x})=\left[Q^A(x),V\left(x^0\right)\right]\Big|_{Q^A{}_0(\mathbf{x})},\end{array}\right. \tag{29}$$

where ρ_{AB} *is a constant, symmetric, and invertible matrix.*

Theorem 4 extends the result of Theorem 3 to field theories.

Examples

E_1) For the Schrödinger Lagrangian with a time-independent potential, the second-order Lagrangian $\hat{\mathcal{L}}_0$ takes the form

$$\hat{\mathcal{L}}_0=\lambda\hbar\mathcal{H}^1\mathcal{H}^2,\ \lambda=\mathrm{cons\tan t}, \tag{30}$$

with

$$\mathcal{H}^A\equiv\dot{Q}^A-\frac{(-)^A}{\mathrm{i}\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-U(\mathbf{x})\right)Q^A,\ Q^A=(\psi,\psi^*),\ A=1,2. \tag{31}$$

E_2) For the Dirac field, the Lagrangian $\hat{\mathcal{L}}_0$ reads as

$$\hat{\mathcal{L}}_0=\theta\left(\partial_\mu\bar{\psi}_a\partial^\mu\psi^a+m^2\bar{\psi}_a\psi^a-2\mathrm{i}m\bar{\psi}_a\left(\gamma^\mu\right)^a{}_b\,\partial_\mu\psi^b\right),\ \theta=\mathrm{cons\tan t}. \tag{32}$$

By direct computation we find that the relationship between the functions E^A and $\hat{E}_A$ (see equations (9) and (28)) is given by

$$\hat{E}_A+k_{AB}E^B-\left(k_{AC}\frac{\partial\left[z^C,V\right]}{\partial z^B}-k_{BC}\frac{\partial\left[z^C,V\right]}{\partial z^A}\right)H^B=0, \tag{33}$$

where the matrix k_{AB} from (33) is constant, symmetric, and invertible and, moreover, fulfills no additional requirements. Formulas (33) toghether with the previous results lead to equivalence

$$\left\{\begin{array}{c}\hat{E}_A=0,\\ z^A(t_0)=z_0^A,\\ \dot{z}^A(t_0)=\left[z^A,V\right]\Big|_{z_0^A},\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}E^A=0,\\ z^A(t_0)=z_0^A,\\ \dot{z}^A(t_0)=\left[z^A,V\right]\Big|_{z_0^A},\end{array}\right., \tag{34}$$

that puts on equal footing the equations

$$E^A \equiv \ddot{z}^A - \left[\left[z^A, V\right], V\right] = 0, \tag{35}$$

$$\begin{aligned} \hat{E}_A \quad \equiv \quad & -k_{AB}\ddot{z}^B + \left(k_{AC}\frac{\partial\left[z^C, V\right]}{\partial z^B} - k_{BC}\frac{\partial\left[z^C, V\right]}{\partial z^A}\right)\dot{z}^B + \\ & +k_{BC}\left[z^B, V\right]\frac{\partial\left[z^C, V\right]}{\partial z^A} = 0, \end{aligned} \tag{36}$$

in the presence of the initial conditions

$$z^A\left(t_0\right) = z_0^A,\ \dot{z}^A\left(t_0\right) = \left.\left[z^A, V\right]\right|_{z_0^A}. \tag{37}$$

A similar equivalence is enabled in the case of field theories.

3 Conclusion

The main conclusion of this thesis can be synyhesized into: a) first-order second-class systems endowed with a non-degenerate symplectic 2-form allow (from the classical dynamics viewpoint) for a non-degenerate second-order Lagrangian formulation that is equivalent to the first-order formulation (in terms of the same variables) at the level of the solutions to the equations of motion subject to some mutually compatible initial conditions; b) the number of physical degrees of freedom associated with the second-order formulation is twice the similar number corresponding to the linear formulation.

As a consequence, although the solutions to the Cauchy problems related to the two formulations coincide, this is no longer valid with respect to their number of physical degrees of freedom.

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