Summary on doctoral thesis
CLASSES OF DEDUCTIVE SYSTEMS IN BCK-ALGEBRA

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1 Introduction

The aim of this thesis is to present new results in the field of residuated lattices, specifically by introducing and studying the notion of *implicative ideals* (*i-ideals*, for short) in residuated lattices, as well as studying several classes of residuated lattices as *noetherian* and *artinian*.

This situates the subject of this thesis in the field of the algebra of logic. The results we are presenting in this thesis come from the following original papers belonging to the author of this thesis: [19], [20]and [52]. In this introduction, we discuss our motivation for studying these topics, and give an overview of the chapters to come.

The theory of residuated lattices was used to develop algebraic counterparts of fuzzy logics ([58]) and substructural logics ([51]).

A *residuated lattice* ([30],[66]) is an algebraic structure $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $(L, \odot, 1)$ is a commutative monoid and, for all $x, y, z \in L$,

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z.$$

In order to simplify the notation, a residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ will be referred by its support set L, as will be the case for any algebraic structure we will be working with. By $B(L)$ we denote the set of all *complemented* elements of L.

In 2000, R. Cignoli et al. ([23]) proved that Hájek's logic is the logic of continuous t-norms as conjectured by P. Hájek. A recent study on BL-algebras was realized in 2012, by S. Motamed and J. Maghaderi ([46]), they introduce the notions of *noetherian* and *artinian* BL-algebras as BL-algebras with the property that every increasing (decreasing) chain of filters is stationary. They obtained some equivalent definitions of noetherian and artinian BL-algebras and proved the Anderson and Cohen theorems of rings theory ([37]) in BL-algebras. They proposed an open problem about noetherian (artinian) BL-algebras and short exact sequences in BL-algebras. Soon after, in 2013, O. Zahiri ([67]) solved the open problem appeared in S. Motamed and J. Maghaderi ([46]) about the relation between noetherian (artinian) BL-algebras in short exact sequences, and some improvements have been made. We must notice that the theory of noetherian (artinian) algebras in the general case of posets was studied in 1983, by C. Năstăsescu ([50]), in the romanian scientific literature. One of our goals in this thesis is to study noetherian and artinian algebras in the general case of residuated lattices.

The thesis consists of five chapters and is organized as follows: In Chapter 2 we recall the basic definitions and we put in evidence examples of residuated lattices and many rules of calculus which we need in the rest of the thesis and a suggestive diagram of the classification of distributive residuated lattices. Also, we present a study on implicative filters (i-filter, for short) theory in residuated lattices. We mention some samples. We focus our study on properties of implicative filters in residuated lattices. We are interested on finitely generated i-filters and to investigate under which condition the set $Min_L(F)$ of all minimal prime i-filters belonging to the i-filter F of L is a finite set (see Theorem 2.5).

Theorem 2.5. Let $F \in \mathcal{F}_i(L)$ a proper i-filter. If every element of $Min_L(F)$ is finitely generated, then $Min_L(F)$ is a finite set.

Also, we investigate the existence of an i-filter in the direct product residuated lattice $L \times L^{'}$ (see Theorem 2.6).

Theorem 2.6. Let $L, L^{'}$ be residuated lattices. Then K is an i-filter of $L \times L^{'}$ iff there exist $F \in \mathcal{F}_i(L)$ and $G \in \mathcal{F}_i(L^{'})$ such that $K = F \times G$.

In Chapter 3, The filter theory of the logical algebras plays an important role in studying these algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Sometimes, a filter is also called deductive system ([48]). At present, the filter theory of residuated lattices has been studied and some important results have been obtained ([8], [9], [61], [68], [69]).

In [8] we proposed a new approach for the study of filters in residuated lattices.

In this paper we work in the general case of residuated lattices and we establish some relationships between regular filters and other filters: Boolean filters, MV filters, Stonean filters, divisible filters and by some examples we show that these filters are different.

Also, we prove that a residuated lattice L is a Stonean residuated lattice if and only if every filter of L is a Stonean filter and we show that a residuated lattice L has the Double Negation Condition if and only if every filter is a regular filter.

In Chapter 4, we realize a study of noetherian and artinian algebras in the general case of residuated lattices. In 1983, C. Năstăsescu ([50]) had realized a study on posets with the property that every decreasing (increasing) chain of elements of a poset A is stationary, based on this study we define a *noetherian* (*artinian*) *poset* A as a poset with the property that every decreasing (increasing) chain of elements of A is stationary (see Remark 4.2). Also, our study is concerned on noetherian and artinian algebras in the case of modular lattices (see Theorem 4.2, Remark 4.1, Remark 4.2).

Theorem 4.2. Suppose that A is a bounded modular lattice. Then A is a noetherian and artinian lattice iff A has a composition sequence.

Remark 4.1. If A is a modular lattice and $a \in A$, then A is a noetherian (artinian) lattice iff the lattices $(a]$ and $[a)$ are noetherian (artinian).

Remark 4.2. Suppose that A is a bounded modular lattice. Then A is a noetherian and artinian lattice iff A has a composition sequence.

Some of these results are transferred to residuated lattices as simple consequences (see Corollary 4.2).

Corollary 4.2. If L is a residuated lattice, then L is a noetherian and artinian residuated lattice iff in the lattice $(F(L), \subseteq)$ there is a composition sequence $\{1\} = F_0 \subseteq F_1 \subseteq ... \subseteq F_n = L$.

Recently, S. Motamed and J. Maghaderi ([46]) introduced the notions of *noetherian* and *artinian* BL-algebras as BL-algebras with the property that every increasing (decreasing) chain of filters is stationary. In this way for a BL-algebra L all the properties from the theory of noetherian (artinian) posets are transferred to the lattice $(\mathcal{F}(L), \subseteq)$ of all i-filters of L. They obtained some equivalent definitions of noetherian and artinian BL-algebras and proved the *Anderson* and *Cohen* theorems of rings theory ([37]) in BL-algebras. They proposed an open problem about noetherian (artinian) BL-algebras and short exact sequences in BL-algebras. Soon after, in 2013, O. Zahiri ([67]) solved the open problem appeared in S. Motamed and J. Maghaderi ([46]) about the relation between noetherian (artinian) BL-algebras in short exact sequences, and some improvements have been made.

Using the model of posets ([50]), rings ([37]) and BL-algebras ([46],[67]), we introduce the notion of *noetherian* and *artinian* in the general case of residuated lattices.

Definition 4.4. A residuated lattice L is called *noetherian (artinian)* if the poset $(\mathcal{F}_i(L), \subseteq)$ is noetherian (artinian).

We present some equivalent definitions of noetherian and artinian residuated lattices and proved the *Anderson* and *Cohen* theorems of ring theory in residuated lattices (see Theorem 4.7).

Theorem 4.7. L is a noetherian residuated lattice iff every prime i-filter of L is finitely generated.

Also, after we define the notion of *weak short exact sequence* in residuated lattices (see Definition 2.10) as a short exact sequence $L \xrightarrow{f} L^{'} \xrightarrow{g} L^{''}$ with the property that $Ker(g) \subseteq Im(f)$, we solve in the general case of residuated lattices an open problem about the relation between noetherian (artinian) residuated lattices in short exact sequences (see Theorem 4.13), that was initially proposed for the case of BL-algebras in [46].

Theorem 4.13. Let $L \xrightarrow{f} L^{'} \xrightarrow{g} L^{''}$ be a weak exact sequence of residuated lattices such that f is one-to-one and g is onto. Then $L^{'}$ is a noetherian (artinian) residuated lattice iff L and $L^{''}$ are noetherian (artinian) residuated lattices.

Finally, we study the direct product of two noetherian (artinian) residuated lattices (see Theorem 4.12).

Theorem 4.12. Let L_1 and L_2 be residuated lattices. Then L_1 and L_2 are noetherian (artinian) residuated lattices iff $L_1 \times L_2$ is a noetherian (artinian) residuated lattice.

In Chapter 5 , the reticulation was first defined for commutative rings by Simmons in [56] and it was extended by Belluce to non–commutative rings in [3].

The reticulation of a ring R is a bounded distributive lattice $L(R)$ such that the prime spectrum of R, endowed with the Zariski topology, is homeomorphic to the prime spectrum of $L(R)$, endowed with the Stone topology. By this connection many properties can be transported from R to $L(R)$ and vice versa. Hence, a natural problem is to define a reticulation for others classes of universal algebra.

This was done by Belluce for MV-algebras in [2], Georgescu for quantales in [31] (which constitute a good abstraction of the lattice of congruences for many types of algebraic structures), Leuştean for BL-algebras in [44], Mureşan for residuated lattices in [47], [48] and Buşneag and Piciu [13] for Hilbert algebras.

So, generally speaking, the reticulation for an algebra A of types mentioned above is a pair (L_A, λ) consisting of a bounded distributive lattice L_A and a surjection $\lambda : A \to L_A$ so that the function given by the inverse image of λ induces (by reticulation) a homeomorphism of topological spaces between the prime spectrum of L_A and that of A. This construction allows many properties to be transferred between L_A and A.

Using the model of the above structures, in this paper we will define the reticulation of a bounded BCK algebra and prove several properties of it.

The paper is divided in seven sections. In the first three sections we recall all the preliminary notations and results relative to BCK algebras and relevant to the paper.

In Section 4 for any BCK algebras A, the family $\tau_A = \{D(X) : X \subseteq A\}$ is a topology on $Spec(A)$, having $\{D(a) : a \in A\}$ a basis. The topology τ_A is called the Zariski topology on $Spec(A)$ and the topological space $(Spec(A), \tau_A)$ is called the prime spectrum of A. In the sequel, let A be a BCK algebra and for $a \in A$ we define $D_{Max(A)}(a) = D(A) \cap Max(A) = \{M \in Max(A) : a \notin M\}$.

We have two results:

1. Let A be a bounded BCK algebra. The for any $a \in A, D_{Max(A)}(a)$ is a compact set in $Max(A)$ (Proposition 5.7).

2. If A is a bounded BCK algebra, the $Max(A)$ is a compact Hausdorff topological Space (Theorem 5.10).

In Section 5 will define a new bounded distributive lattice L_A will be called Belluce lattice associated with A.

The principal result is:

If A is a bounded BCK algebra, the $u_A : Max(A) \to Max(L_A)$ is a homeormorphism beteen the topological Spaces $Max(A)$ and $Max(L_A)$ (Theorem 5.11).

In Section 6 we define the notion of *reticulation* of a bounded BCK algebra A (Definition 5.9) and prove the uniqueness of the reticulation (Theorem 5.12).

In Section 7 we define the notion of normal bounded BCK algebra (using the model of lattices) and prove that if A is a normal bounded BCK algebra, then L_A is normal lattice (Corollary 5.1).

2 Preliminaries

In this chapter we present the basic definitions and properties on integral commutative residuated lattices (residuated lattices, for short) and we put in evidence many rules of calculus in a residuated lattice which we need in the rest of the thesis.

Most of the results in this chapter are known from previous works; however, the results that come with proofs are original, unless mentioned otherwise; their basic nature or their simplicity in some cases has made them suitable for presentation in this introductive chapter.

One of the topics studied in this chapter is the distributivity of a residuated lattice. However, the necessary and sufficient condition for distributivity in a residuated lattice remains an open subject. Another topic studied is the filter theory in residuated lattices. We mention some examples. We investigate under which condition the set $Min_L(F)$ of all minimal prime i-filters belonging to the i-filter F of L is a finite set (see Theorem 2.5). Also, we investigate the existence of an i-filter in the direct product residuated lattice $L \times L^{'}$ (see Theorem 2.6). Finally, we recall some notions on morphisms of residuated lattices and we introduce the notion of *weak short exact sequence* for residuated lattices (see Definition 2.10).

3 Notes on regular filters in residuated lattices

In the mathematical literature, many types of filters in residuated lattices have been studied. In [8] we proposed a new approach for the study of these filters. In this paper, in the spirit of [8], we establish some connections between regular filters and other types of filters in residuated lattices. Also, we prove that a residuated lattice is a Stonean residuated lattice if and only if every filter is a Stonean filter and we show that a residuated lattice has the Double Negation Condition if and only if every filter is a regular filter.

4 Noetherian and artinian algebras in the general case of residuated lattices

Using the model of ordered sets ([50]), rings ([37]) and BL-algebras ([46],[67]), in this chapter we introduce the notion of *noetherian* and *artinian* in the general case of residuated lattices, as residuated lattices with the property that every increasing (decreasing) chain of filters is stationary (see Definition 4.4). In Example 8, (*ii*) we present a residuated lattice which is not noetherian, hence the study on noetherian residuated lattices is proper to be done. Several interesting results are obtained. We mention some samples. We obtain some equivalent definitions of noetherian and artinian residuated lattices (see Theorems 4.3 and 4.5). We study the quotient residuated lattice of a noetherian (artinian) residuated lattice (see Remark 4.3) and proved the *Anderson* and *Cohen* theorems of ring theory ([37]) in residuated lattices (see Theorem 4.7). After we introduce the notion of *weak short exact sequence* for residuated lattices (see Definition 2.10), we solve in the general case of residuated lattices about the relation between noetherian (artinian) residuated lattices in short exact sequences (see Theorem 4.13), that was initially proposed for the case of BL-algebras in [46]. Finally, we study the direct product of two noetherian (artinian) residuated lattices (see Theorem 4.12).

5 The Belluce-lattice associated with a bounded BCK-algebra

This chapter is structured into five sections as follows: contains the some basic definitions and results regarding BCK algebras. The first one contains deductive systems of a BCK algebras. For any BCK algebras A, the topological spaces Spec(A) and Max(A) is studied in section 2. Section 3 will define a new bounded distributive lattice L_A will be called Belluce lattice associated with A. In section 4 we define the notion of reticulation of a bounded BCK-algebra A. In section 5 we define the notion of a normal bounded BCK-algebra.

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