Universitatea din Craiova
Şcoala Doctorală de Ştiinţe
Domeniul Matematică

Teză de doctorat:

Methods of nonlinear analysis in the study of elliptic problems

Summary

Stîrcu Iulia Dorotheea

Conducător de doctorat:
Prof. Univ. Dr. *Vicenţiu Rădulescu*

Craiova
2018

Summary

This work is mainly concerned with the study of nonlinear problems associated with some elliptic partial differential operators. Our goal is to explore the existence of weak solutions for certain PDE's of elliptic type.

According to V. Rădulescu and D. Repovš in [15], *"Partial differential equations are a precise, elegant, rich, and captivating subject, which is quite old, and its history is broad and deep. Elliptic partial differential equations are startling due to their elegance and clarity."* The first partial differential equation was the vibrating chord equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ represents the elongation at point x and at time t, and the positive constant a represents the ratio between the constant pressure exerted on the chord and its density.

Initially, the study was focused on linear equations with partial derivatives, such as the Laplace, Poisson and Helmholz equations. Very rapidly they appeared numerous results concerning nonlinear elliptic problems and the qualitative analysis of their solutions. Up to the present, the study of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, elasticity, optics, etc) has continued to be one of the fundamental concerns of the development of partial differential equations.

The idea of using analytic methods to study partial differential equations was started for the first time by H. Poincaré [10].

One of the main ideas in searching (weak) solutions for partial differential equations is focused on the critical point theory. An equation can be associated with an energy functional whose critical points represent the solutions of the equation. We can easily obtain critical points for a functional I by looking for a *minimum* of I which is to be attained as limit of a minimizing sequence.

Another way to finding critical points for a functional I is the *mountain pass lemma* of A. Ambrosetti and P. H. Rabinowitz [1]: if the functional I is a C^1 function on a Banach space X, satisfying the Palais-Smaile condition (i.e., any sequence $u_n \in X$ such that $I(u_n)$ is uniformly bounded and $I'(u_n) \to 0$ has a strongly convergent sub-sequence in X) and the following geometric conditions

i) $I(0) = 0$

ii) $I(u) \geq a > 0$, for all $u \in X$ with $\|u\| = R$

iii) $I(u_0) \leq 0$, for some $u_0 \in X$ with $\|u\| > R$

then there exists a nontrivial critical point v of I such that $I'(v) = 0$ and $I(v) \geq a$.

We make in what follows a brief description of this thesis.

Chapter 2, *Function spaces*, represents a description of a variable exponent Lebesgue-Sobolev spaces and Orlicz-Sobolev spaces wich are used in the study of many problems that will be presented in the following chapters.

Chapter 3 is based on the paper *Characterization of solutions to equations involving p(x)-Laplace operator* published in *Electronic Journal of Differential Equations* (see Reference [19]). In this article we establish two results. The first one proves an alternative for a nonlinear eigenvalue problem involving the $p(x)$-Laplacian. Several ideas developed in the study of the spectrum of such general operators in divergence form are developed by Mihăilescu, Rădulescu, Repovš in [6], Molica Bisci, Repovš in [7], and Stăncuţ, Stîrcu in [16]. In the second part we study an existence result of a subcritical boundary value problem for the same operator. To prove our first result we use a mountain pass lemma on the product space $W_0^{1,p(\cdot)}(\Omega) \times \mathbb{R}$, considering a special hyperplane which is intended to separating surface instead of a sphere [8, 14]. The result obtained in the second problem is based on a special version of the mountain pass lemma of Ambrosetti-Rabinowitz [9].

Let Ω be a bounded domain in $\mathbb{R}^N$. In the first part of this chapter we are concerned in the study of the following nonlinear eigenvalue problem involving the $p(x)$-Laplacian

$$\begin{gathered} -\Delta_{p(x)} u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ 0 < \lambda \leq a, \end{gathered} \tag{1}$$

with constraints on eigenvalues, where a is a positive constant, p satisfies the following properties:

$$\begin{gathered} p \in C_+(\overline{\Omega}), \\ 1 < p^- \leq p(x) \leq p^+ < \infty, \\ p \text{ is global log-Hölder continuous} \end{gathered} \tag{2}$$

and the function f satisfies the following conditions

(H1) f is a measurable function in $x \in \Omega$ and continuous in $u \in \mathbb{R}$, with $f(x,0) \neq 0$ on a subset of Ω (where $|\Omega| > 0$); then, f is a *Carathéodory* function;

(H2) $|f(x,u)| \leq c_1 + c_2|u|^{q(x)-1}$ for almost everywhere in Ω and all $u \in \mathbb{R}$, where c_1 and c_2 are two positive constants, $q \in C_+(\overline{\Omega})$ and $1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < p^*(x)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

(H3) for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$, there exist $b_1 \geq 0$ and $b_2 \geq 0$ two constants, β a continuous function and ν a constant with $1 \leq \beta(x) < p(x) < \nu$ such that

$$f(x,u)u - \nu \int_0^u f(x,t)dt \geq -b_1 - b_2|u|^{\beta(x)}.$$

Then is natural to look for weak solutions of this kind of problems in the variable exponent Sobolev spaces.

Definition 1. *We say that* $u \in W_0^{1,p(x)}(\Omega)$ *is a* weak solution *of problem* (1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx - \lambda \int_\Omega f(x,u)\varphi\, dx = 0, \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega).$$

The main result concerning the nonlinear eigenvalue problem (1) is the following:

Theorem 1 ([19])**.** *Suppose that relation* (2) *holds and the hypotheses* (H1)–(H3) *are satisfied by the function* $f : \Omega \times \mathbb{R} \to \mathbb{R}$. *Let* $\gamma : \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function such that, for some constants* $0 < \rho < r$, $\sigma > 0$, *the following relations hold:*

(1) $\gamma(0) = \gamma(r) = 0$;

(2) $\gamma(\rho) = \dfrac{a_1 + a_2}{\sigma + 1}$;

(3) $\lim_{|t|\to\infty} \gamma(t) = +\infty$;

(4) $\gamma'(t) < 0$ *if and only if* $t < 0$ *or* $\rho < t < r$.

Then, for every $a > 0$, *one of the following alternatives holds:*

(a) for the problem (1), $a > 0$ *is an eigenvalue with the corresponding eigenfunction* $u \in W_0^{1,p(x)}(\Omega)$ *established by*

$$\alpha \le -\int_\Omega \int_0^{u(x)} f(x,t)\, dt\, dx + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx \le a_1 + \alpha$$

or

(b) one can state $z > 0$ *a number which satisfies*

$$\rho < z < r \tag{3}$$

and determines by means of the following relations an eigensolution $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times (0, a]$ *of the problem* (1)*:*

$$\|u\| = |z|^{-\sigma/q^-} \left(-\gamma'(z)\right)^{1/q^-} \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{-1/q^-}, \tag{4}$$

$$\lambda^{-1} = z\big(-\gamma'(z)\big)\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)^{-1} + a^{-1}, \tag{5}$$

$$\begin{aligned} \alpha \le z^{\sigma+1}\|u\|^{q^-} \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + (\sigma+1)\gamma(z) \\ - \int_\Omega \int_0^{u(x)} f(x,t)\, dt\, dx + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx \le a_1 + \alpha. \end{aligned} \tag{6}$$

In order to prove Theorem 1 we state a version of the Mountain-Pass Theorem by Ambrosetti and Rabinowitz.

Lemma 1 ([8]). *Let X be a Banach space and let $J \in C^1(X \times \mathbb{R}, \mathbb{R})$ be a functional satisfying the hypotheses:*

(i) there exist $\rho > 0$ and $\alpha > 0$ two constants such that $J(v, \rho) \geq \alpha$, for every $v \in X$;

(ii) there exists $r > \rho$ with $J(0,0) = J(0,r) = 0$. Then we have a critical value of J, denoted by

$$c := \inf_{g \in \Gamma} \max_{0 \leq \tau \leq 1} J(g(\tau)),$$

where

$$\Gamma = \{g \in (C([0,1]), X) \times \mathbb{R}; g(0) = (0,0), g(1) = (0,r)\}$$

and

$$c \geq \inf_{v \in X} J(v, \rho) \geq \alpha > 0.$$

The second problem studied in Chapter 3 is the following

$$\begin{aligned} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2} u + |u|^{q(x)-2} u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \\ u \not\equiv 0 \quad & \text{in } \Omega, \end{aligned} \tag{7}$$

where $\Omega \subset \mathbb{R}^N (N > 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, p, q are continuous functions on $\overline{\Omega}$ which satisfy

$$1 < p(x) < q(x) < p^*(x),$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$, for all $x \in \overline{\Omega}$.

Definition 2. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (7) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega |u|^{p(x)-2} u \varphi \, dx + \int_\Omega |u|^{q(x)-2} u \varphi \, dx,$$

for every $\varphi \in W_0^{1,p(x)}(\Omega)$.

The following theorem represents our existence result.

Theorem 2 ([19]). *If $\lambda < \lambda_{P^*}$, where*

$$\lambda_{P^*} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)} \, dx}{\int_\Omega |u|^{p(x)} \, dx},$$
$$1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < p^*(x),$$

with p satisfying hypothesis (2), then there exists a weak solution for the problem (7).

The main tool that we use in the proof of the second result is the Mountain-Pass Theorem in the following variant.

Theorem 3 ([9]). *Let X be a real Banach space and $F \in C^1(X, \mathbb{R})$ be a functional which satisfies the Palais-Smale condition. If F satisfies the following geometric conditions*

(1) there exist two constants $R, c_0 > 0$ such that $F(u) \geq c_0$, for every $u \in X$ with $\|u\| = R$,

(2) $F(0) < c_0$ and there exists $v \in X$ with $\|v\| > R$ such that $F(v) < c_0$, then there exists at least a critical point for the functional F.

Chapter 4 is based on the paper *Eigenvalue problems for anisotropic equations involving a potential on Orlicz-Sobolev type spaces* published in *Opuscula Mathematica* (see Reference [16]).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. Consider that $a_i : (0, \infty) \to \mathbb{R}$, $i \in \{1, ..., N\}$, are functions such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, ..., N\}$, defined by

$$\varphi_i(t) = \begin{cases} a_i(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, $\lambda > 0$ is a real number, $V(x)$ is a potential and q_1, q_2, $m : \overline{\Omega} \to (2, \infty)$ are continuous functions. This chapter is devoted to the study of the anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i(\varphi_i(\partial_i u)) + V(x)|u|^{m(x)-2}u = \lambda(|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $V : \Omega \to \mathbb{R}$ satisfies $V \in L^{r(x)}(\Omega)$ with $r(x) \in C(\overline{\Omega})$.

Considering that the operator in the divergence form is nonhomogeneous we introduce an Orlicz-Sobolev space setting for problems of type (8). In fact, given that our problem contains an equation of anisotropic type, we seek weak solutions in a more general Orlicz-Sobolev type space, namely anisotropic Orlicz-Sobolev space. At the same time we note the presence of the continuous exponent functions m, q_1 and q_2 which leads us to use a suitable variable exponent Lebesgue space setting.

In this chapter we look for weak solutions of problem (8) in a subspace of the anisotropic Orlicz-Sobolev space $W_0^1 L_{\vec{\Phi}}(\Omega)$, namely

$$E := \left\{ u \in W_0^1 L_{\vec{\Phi}}(\Omega);\ \int_\Omega |V(x)||u|^{m(x)} dx < \kappa,\ \text{with } \kappa > 0 \text{ real constant} \right\}.$$

Define the functionals $J_V, I : E \to \mathbb{R}$ by

$$J_V(u) = \int_\Omega \sum_{i=1}^{N} \Phi_i(|\partial_i u|)dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx,$$
$$I(u) = \int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx.$$

The energy functional corresponding to problem (8) is defined as $T_\lambda : E \to \mathbb{R}$,

$$T_\lambda(u) = J_V(u) - \lambda I(u).$$

The main results of the present chapter are given by the following three theorems.

Theorem 4 ([16])**.** *Assume that the functions $q_1, q_2, m \in C(\overline{\Omega})$ satisfy the hypothesis*

$$2 < (P^0)_+ < q_2^- \le q_2^+ \le m^- \le m^+ \le q_1^- \le q_1^+ < q_1^+ \cdot r^{-\prime} < (P_0)^*, \tag{9}$$

where $r^{-\prime} = \frac{r^-}{r^- - 1}$. Then any $\lambda > 0$ is an eigenvalue of problem (8)*.*

Theorem 5 ([16])**.** *Assume that the functions $q_1, q_2, m \in C(\overline{\Omega})$ verify the condition*

$$2 < q_2^- \le q_2^+ \le q_1^- \le q_1^+ \le m^- \le m^+ < m^+ \cdot r^{-\prime} < (P_0)_- \le (P_0)^*, \tag{10}$$

where $r^{-\prime} = \frac{r^-}{r^- - 1}$. Then there is $\lambda_* > 0$ so that every $\lambda \in (0, \lambda_*]$ is an eigenvalue of problem (8)*.*

Theorem 6 ([16])**.** *Assume that the functions $q_1, q_2, m \in C(\overline{\Omega})$ fulfill the hypothesis*

$$2 < q_2^- \le q_2^+ \le m^- \le m^+ \le q_1^- \le q_1^+ < q_1^+ \cdot r^{-\prime} < (P_0)_- \le (P_0)^*, \tag{11}$$

where $r^{-\prime} = \frac{r^-}{r^- - 1}$. Then there is $\lambda^* > 0$ so that every $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of problem (8)*.*

Chapter 5 is based on the paper *An existence result for a quasilinear degenerate problem in $\mathbb{R}^N$* published in *Electronic Journal of Qualitative Theory of Differential Equations* (see Reference [17]). In this chapter we study the problem

$$-\text{div}[\phi'(|\nabla u|^2)\nabla u] + a(x)|u|^{\alpha-2}u = |u|^{\gamma-2}u + |u|^{\beta-2}u \text{ in } \mathbb{R}^N (N \ge 3), \tag{12}$$

where a is a positive potential satisfying the following hypotheses:

(a_1) $a \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$;

(a_2) $\text{essinf}_{\mathbb{R}^N} a > 0$;

(a_3) $\lim_{x\to 0} a(x) = \lim_{|x|\to\infty} a(x) = +\infty$,

$1 < p < q < N$, $1 < \alpha \le p^* q'/p'$ and $\max\{\alpha, q\} < \gamma < \beta < p^* = pN/(N-p)$, with p' and q' the conjugate exponents, respectively, of p and q.

We assume that the function ϕ, which generates the differential operator in problem (12), satisfies the following hypotheses:

(ϕ_1) $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$;
(ϕ_2) $\phi(0) = 0$;
(ϕ_3) there exists $c_1 > 0$ such that

$$\begin{cases} c_1 t^{p(x)/2} \leq \phi(t) \text{ if } t \geq 1, \\ c_1 t^{q(x)/2} \leq \phi(t) \text{ if } 0 \leq t \leq 1; \end{cases}$$

(ϕ_4) there exists $c_2 > 0$ such that

$$\begin{cases} \phi(t) \leq c_2 t^{p(x)/2} \text{ if } t \geq 1, \\ \phi(t) \leq c_2 t^{q(x)/2} \text{ if } 0 \leq t \leq 1; \end{cases}$$

(ϕ_5) there exists $0 < \mu < 1$ such that $2t\phi'(t) \leq \gamma\mu\phi(t)$ for all $t \geq 0$;
(ϕ_6) the mapping $t \longmapsto \phi(t^2)$ is strictly convex.

Our purpose is to prove, by means of the mountain pass theorem (see [11, 12, 13]), that problem (12) admits at last one weak solution.

Now, we define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{L^p+L^q} := \inf\left\{ \|v\|_p + \|w\|_q \, ; v \in L^p(\mathbb{R}^N), w \in L^q(\mathbb{R}^N), u = v + w \right\}. \tag{13}$$

We set $\|u\|_{p,q} = \|u\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)}$.

The property that $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ are Orlicz spaces, as well as others properties of these spaces, has been proved by M. Badiale, L. Pisani and S. Rolando in [5].

In order to use them throughout this chapter, we state the following result that is also found in [4]:

Proposition 1. *Let $\Omega \in \mathbb{R}^N$, $u \in L^p(\Omega) + L^q(\Omega)$. We have:*
(i) if $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^p(\Omega')$;
(ii) if $\Omega' \subset \Omega$ is such that $u \in L^\infty(\Omega')$, then $u \in L^q(\Omega')$;
(iii) $|[|u(x) > 1|]| < +\infty$;
(iv) $u \in L^p([|u(x)| > 1]) \cap L^q([|u(x)| \leq 1])$;
(v) the infimum in (13) is attained;
(vi) $L^p(\Omega) + L^q(\Omega)$ is reflexive and $(L^p(\Omega) + L^q(\Omega))' = L^{p'}(\Omega) \cap L^{q'}(\Omega)$;
(vii) $\|u\|_{L^p(\Omega)+L^q(\Omega)} \leq \max\left\{ \|u\|_{L^p([|u(x)|>1])}, \|u\|_{L^q([|u(x)|\leq 1])} \right\}$;
(viii) if $B \in \Omega$, then $\|u\|_{L^p(\Omega)+L^q(\Omega)} \leq \|u\|_{L^p(B)+L^q(B)} + \|u\|_{L^p(\Omega\setminus B)+L^q(\Omega\setminus B)}$.

Finally, we define the function space

$$X := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| = \|\nabla u\|_{p,q} + \left(\int_{\mathbb{R}^N} a(x)|u|^\alpha dx \right)^{1/\alpha}.$$

We remark that X is continuously embedded in W defined by Azzollini in [4], where

$$W := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

$$\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha .$$

Definition 3. *A weak solution of problem* (12) *is a function* $u \in X \setminus \{0\}$ *such that*

$$\int_{\mathbb{R}^N} \left[\phi^{'}(|\nabla u|^2)(\nabla u \cdot \nabla v) + a(x)|u|^{\alpha-2}uv - |u|^{\gamma-2}uv - |u|^{\beta-2}uv\right] dx = 0,$$

for any $v \in X$.

We define the energy functional $I : X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^{\alpha}dx - \frac{1}{\gamma}\int_{\mathbb{R}^N} |u|^{\gamma}dx - \frac{1}{\beta}\int_{\mathbb{R}^N} |u|^{\beta}dx.$$

Now, we give a version of the mountain pass lemma of A. Ambrosetti and P. Rabinowitz [2] (see also [3]).

Lemma 2. *Let* X *be a Banach space and assume that* $I \in C^1(X, \mathbb{R})$ *satisfies the following geometric hypotheses:*

(a) $I(0) = 0$;

(b) *there exist two positive numbers* a *and* r *such that* $I(u) \geq a$ *for any* $u \in X$ *with* $\|u\| = r$;

(c) *there exists* $e \in X$ *with* $\|e\| > r$ *such that* $I(e) < 0$.

Let

$$P := \{p \in C([0, 1]; X); p(0) = 0, p(1) = e\}$$

and

$$c := \inf_{p \in P} \sup_{t \in [0,1]} I(p(t)).$$

Then there exists a sequence $(u_n) \subset X$ *such that*

$$\lim_{n\to\infty} I(u_n) = c \text{ and } \lim_{n\to\infty} \left\|I^{'}(u_n)\right\|_{X^*} = 0.$$

Moreover, if I *satisfies the Palais-Smale condition at the level* c, *then* c *is a critical value of* I.

Finally, the main result of this chapter is given by the following theorem.

Theorem 7 ([17]). *Suppose that* $1 < p < q < N$, $1 < \alpha \leq p^* q^{'}/p^{'}$, $\max\{\alpha, q\} < \gamma < \beta < p^*$, $(a_1) - (a_3)$ *and* $(\phi_1) - (\phi_6)$ *are satisfied. Then problem* (12) *has at last one weak solution.*

Chapter 6 is based on the paper *An existence result for quasilinear elliptic equations with variable exponents* published in *Annals of the University of Craiova, Mathematics and Computer Science Series* (see Reference [18]). Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ a bounded domain with smooth boundary. Let λ be a positive real parametre, p, r, s continuous functions on $\overline{\Omega}$ which satisfy the condition

$$2 \leq p(x) < r(x) < s(x) < p^*(x), \tag{14}$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \overline{\Omega}$.

In this chapter we study a nonlinear elliptic equations of $p(x)$-Laplace type

$$\begin{cases} -\mathrm{div}(\phi(x,|\nabla u|)\nabla u) + |u|^{p(x)-2}u = \lambda|u|^{r(x)-2}u - h(x)|u|^{s(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{15}$$

where $\phi(x,t)$ is of type $|t|^{p(x)-2}$ satisfying the following hypotheses:

$(h1)$ $\phi : \Omega \times [0,\infty) \to [0,\infty)$ satisfies the following hypotheses: $\phi(\cdot,\omega)$ is a measurable function on Ω for all $\omega > 0$ and $\phi(x,\cdot)$ is locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$;

$(h2)$ Let $a \in L^{p'(x)}(\Omega)$ be a function and b a nonnegative constant such that

$$|\phi(x,|v|)v| \leq a(x) + b|v|^{p(x)-1}, \forall x \in \Omega, \forall v \in \mathbb{R}^N;$$

$(h3)$ There exists $c > 0$ a constant such that, for almost all $x \in \Omega$, the following condition hold:

$$\phi(x,\omega) \geq c\omega^{p(x)-2}, \text{ for almost all } \omega > 0,$$

with continuous function $p : \overline{\Omega} \to (1,\infty)$ and $h : \overline{\Omega} \to [0,\infty)$ is a continuous function which satisfies the following hypotheses:

$$\left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}}\right)^{\frac{1}{s(x)-r(x)}} \in L^1(\Omega), \tag{16}$$

$$\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}} \in L^{\frac{s(\cdot)}{s(\cdot)-2}}(\Omega). \tag{17}$$

Our purpose in this chapter is to establish, under suitable conditions on ϕ, that for λ large enough there exist at least two nontrivial weak solutions. In order to prove this result, we use a special version of the mountain pass theorem (see [1] and [20, Theorem 1.15]) and a corresponding variational method.

Throughout this chapter, we seek weak solutions for problem (15) in a subspace of $W_0^{1,p(\cdot)}(\Omega)$, more exactly in the *weighted variable exponent Sobolev space* defined by

$$X = \left\{ u \in W_0^{1,p(\cdot)}(\Omega); \int_\Omega h(x)|u|^{s(x)}dx < \infty \right\}$$

endowed with the norm

$$\|u\|_X = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)} + |u|_{h,s(\cdot)}.$$

Definition 4. *We call weak solution for problem* (15) *a function* $u \in X$ *which satisfies*

$$\int_\Omega \Big(\phi(x,|\nabla u|)\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi\Big)dx = \lambda\int_\Omega |u|^{r(x)-2}u\varphi dx - \int_\Omega h(x)|u|^{s(x)-2}u\varphi dx,$$

for any $\varphi \in X$.

We define the *energy functional* $I : X \to \mathbb{R}$ by

$$I(u) = \int_\Omega \Phi(x, |\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \lambda\int_\Omega \frac{1}{q(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{s(x)}dx,$$

where

$$\Phi(x,t) = \int_0^t \phi(x,\omega)\omega d\omega.$$

Finnaly, we state our main result.

Theorem 8 ([18]). *There exists $\lambda_* > 0$ such that for $\lambda > \lambda_*$ problem* (15) *has at least two nontrivial weak solutions.*

In chapter 7 we study the following eigenvalue problem

$$\begin{cases} -\text{div}\,[a(x)(\phi(x,|\nabla u|)\nabla u + \psi(x,|\nabla u|)\nabla u)] = \lambda|u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{18}$$

where $\Omega \in \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, λ is a positive real number, $a : \Omega \to [0,\infty)$ a weighted function which has the property that $a \in L^1_{loc}(\Omega)$ and $p_1, p_2, q \in C_+(\bar{\Omega})$ satisfying

$$1 < p_1(x) < q^- \leq q^+ < p_2(x) < p_1^*(x), \tag{19}$$

where $p_1^*(x) := \dfrac{Np_1(x)}{N - p_1(x)}$ if $p_1(x) < N$ and $p_1^*(x) := +\infty$ if $p_1(x) > N$.

Problem (18) is based on non-homogeneous operators of the type $(\phi(x,|\nabla u|)\nabla u)$. When $\phi(x,\mu) = \mu^{p(x)-2}$, the operator implicated in (18) is the $p(x)$-Laplacian, that is,

$$\Delta_{p(x)}u = \text{div}(|\nabla u|^{p(x)-2}\,\nabla u).$$

Let be the functions $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ which fulfill the following assumptions:

(h_1) $\phi(\cdot,\mu)$ and $\psi(\cdot,\mu)$ are two measurable mappings on Ω for any $\mu \geq 0$; further, $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost $x \in \Omega$;

(h_2) there exist $\alpha_1 \in L^{p_1^{'}}(\Omega)$, $\alpha_2 \in L^{p_2^{'}}(\Omega)$ functions and $\beta > 0$ such that

$$|\phi(x,|v|)v| \leq \alpha_1(x) + \beta|v|^{p_1(x)-1},\;\; |\psi(x,|v|)v| \leq \alpha_2(x) + \beta|v|^{p_2(x)-1}$$

for almost all $x \in \Omega$ and for any $v \in \mathbb{R}^N$.

(h_3) there exists a positive constant $c > 0$ such that

$$\phi(x,\mu) \geq c\mu^{p_1(x)-2},\;\; \phi(x,\mu) + \mu\frac{\partial\phi}{\partial\mu}(x,\mu) \geq c\mu^{p_1(x)-2}$$

and

$$\psi(x,\mu) \geq c\mu^{p_2(x)-2},\;\; \psi(x,\mu) + \mu\frac{\partial\psi}{\partial\mu}(x,\mu) \geq c\mu^{p_2(x)-2},$$

for almost all $x \in \Omega$ and for all positive μ.

Definition 5. *A weak solution for problem* (18) *is a function* $u \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ *such that*

$$\int_\Omega a(x) \left[\phi(x, |\nabla u|) + \psi(x, |\nabla u|)\right] \nabla u \nabla v dx - \lambda \int_\Omega |u|^{q(x)-2} u v dx = 0,$$

for all $v \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}$.

The function space for problem (18) is $D_0^{1,p_2(x)}(\Omega)$, this choice being motivated by hypothesis (19) and the presence of a weight function $a : \Omega \to [0, \infty)$ which satisfies $a \in L^1_{loc}(\Omega)$.

For any $\lambda > 0$ we define $F_\lambda : D_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ by

$$F_\lambda(u) = \int_\Omega \frac{a(x)}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{a(x)}{p_2(x)} |\nabla u|^{p_2(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx.$$

Then, $F_\lambda \in C^1(D_0^{1,p_2(x)}(\Omega), \mathbb{R})$ and

$$\left\langle F_\lambda'(u), v \right\rangle = \int_\Omega a(x) \left[\phi(x, |\nabla u|) + \psi(x, |\nabla u|)\right] \nabla u \nabla v dx - \lambda \int_\Omega |u|^{q(x)-2} u v dx,$$

for all $u, v \in D_0^{1,p_2(x)}(\Omega)$.

We define the *first Rayleigh quotient* by

$$\lambda_1 := \inf_{u \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \frac{a(x)}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{a(x)}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\displaystyle\int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx}.$$

The main result of this chapter is given by the following theorem.

Theorem 9. *Suppose that hypothesis* (19) *is satisfied. Then* $\lambda_1 > 0$. *Furthermore, any* $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem* (18). *Moreover, there exists a positive constant* λ_0 *such that* $\lambda_0 \leq \lambda_1$ *and no* $\lambda \in (0, \lambda_0)$ *is an eigenvalue of problem* (18).

Bibliography

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349–381.

[2] J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, *J. Math. Anal. Appl.*, **306** (2005), 604–618.

[3] N. Chorfi and V. D. Rădulescu, Standing waves solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electronic Journal of the Qualitative Theory of Differential Equations* **37** (2016), 1–12.

[4] X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, *Nonlinear Anal.*, **52** (2003), 1843–1852.

[5] X. Fan, Q. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problems, *J. Math. Anal. Appl.*, **302** (2005), 306–317.

[6] M. Mihăilescu, V. Rădulescu, D. Repovš, On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting, *J. Math. Pures Appl.*, (9) **93** (2010), no. 2, 132–148.

[7] G. Molica Bisci, D. Repovš, Multiple solutions for elliptic equations involving a general operator in divergence form, *Ann. Acad. Sci. Fenn. Math.*, **39** (2014), no. 1, 259–273.

[8] D. Motreanu, A new approach in studying one parameter nonlinear eigenvalue problems with constraints, *Nonlinear Anal.*, **60** (2005), No. 3, 443–463.

[9] D. Motreanu, V. Rădulescu, Existence theorems for some classes of boundary value problems involving the p-Laplacian, *PanAmerican Math. Journal*, **7** (1997), No. 2, 53–66.

[10] H. Poincaré, Sur les équations aux dérivées partielles de la physique mathématique, *Amer. J. Math* **12** (1890), 211–294.

[11] P. Pucci, V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, *Boll. Un. Ital. B*, Ser IX, **III** (2010), 543–582.

[12] P. Pucci, J. Serrin, Extensions of the mountain pass theorem, *J. Funct. Anal.* **59** (1984), 185–210.

[13] P.Pucci, J. Serrin, A mountain pass theorem, *J. Differential Equations* **60** (1985), 142–149.

[14] P. H. Rabinowitz, Minmax Methods in Critical Point Theory with Applications to Differential Equations, *CBMS Reg. Conf. Ser. Math* **65**, Amer. Math. Soc., Providence, R.I., 1986.

[15] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor and Francis Group, Boca Raton FL, 320, pp. 2015.

[16] I. Stăncuţ, I. Stîrcu, Eigenvalue problems for anisotropic equations involving a potential on Orlicz-Sobolev type spaces, *Opuscula Math.*, **36** (2016), no. 1, 81–101.

[17] I. Stîrcu, An existence result for a quasilinear degenerate problem in $\mathbb{R}^N$, *Electronic Journal of Qualitative Theory of Differential Equations*, 2017, No. 5, 1–11.

[18] I. Stîrcu, An existence result for quasilinear elliptic equations with variable exponents, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **44**(2), 2017, 299–315.

[19] I. Stîrcu, V. Uţă, Characterization of solutions to equations involving the $p(x)$-Laplace operator, *Electronic Journal of Differential Equations*, Vol. **2017** (2017), No. 273, pp. 1–16.

[20] M. Willem, *Minimax Theorems*, Birkhäuser, Boston (1996).