Summary of Ph.D. thesis

"Variational and monotonicity methods in the study of nonlinear equations",

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The Ph.D. thesis is mainly concerned with partial differential equations (PDE's). The study of PDE's has its origin in the eighteenth century and it was inspired by concrete models of mechanics (elasticity, gravitational field). Later, this study was also spurred by other physical or chemical problems (e.g., problems of diffusion theory, electrostatic, electricity or magnetism), biological problems, applied mathematics and engineering problems. The first studied partial differential equation was the vibrating string equation

$$\frac{\partial^2 u}{\partial t^2} = a\,\frac{\partial^2 u}{\partial x^2},$$

where $u = u(x,t)$ represents the elongation at point x at time t, while the positive constant a denotes the ratio of constant pressure on the cord and its density.

Initially, the problems involving partial differential equations were linear. Subsequently, differential geometry problems have given rise to nonlinear partial differential equations such as Monge–Ampère equation or minimal surface equation. The study of partial differential equations was also spurred by the classical theory of calculus of variations (based on Euler–Lagrange principle) and Hamilton–Jacobi theory. Moreover, PDE's interacts with a variety of branches of mathematics or physics (such as real analysis, functional analysis, algebraic geometry, mathematical physics or chaos theory).

The principal problem which occurs in the study of PDE's is the existence of solutions. One of the main ideas in searching *weak solutions* for PDE's is based on the *Critical Point Theory.* An equation can be associated with an *energetic functional* whose *critical points* provide the solutions of the equation.

An important tool in finding critical points for a functional Φ is the *Direct Method in the Calculus of Variations* due to Struwe. The idea is to look for a *minimum* of Φ which is to be obtained as limit (in some appropriate sense) of a *minimizing sequence*.

But sometimes it is not possible to minimize a nonnegative continuous function Φ on a complete metric space. An alternative approach is the use of *Ekeland's Variational Principle*. This valuable tool asserts that for a nonnegative C^1 function Φ on a Banach space there is always a minimizing sequence (u_n) so as $\Phi'(u_n) \to 0$.

In the situation when the Direct Method in the Calculus of Variations and Ekeland's Variational Principle fail to be applied, then the *Mountain Pass Theorem* of Ambrosetti and Rabinowitz is another useful result in the Critical Point Theory. This result states that if Φ is a C^1 function on a Banach space X satisfying the *Palais–Smale condition*

(PS) if one has a sequence (u_n) in a manifold M such that $|\Phi(u_n)|$ is bounded and $\Phi'(u_n) \to 0$, then (u_n) is relatively compact (contain a convergent subsequence) in M,

and the *geometric conditions*

$$\Phi(0) = 0,$$

$$\Phi(v) \geq \alpha > 0, \quad \text{for any } v \in X \text{ with } \|v\| = R,$$

$$\Phi(v_0) \leq 0, \quad \text{for some } v_0 \in X \text{ with } \|v_0\| > R,$$

then there exists a nontrivial critical point u of Φ, i.e. $\Phi'(u) = 0$ and $\Phi(u) \geq \alpha$.

But there are also other ways to find solutions to PDE's, for instance we have at hand variants of the above-mentioned methods, such as the *mountain pass theorem without Palais-Smale condition* or the *symmetric mountain pass theorem*.

The aim of present thesis is to explore the existence of weak solutions for certain PDE's. The work contains an introductory part and six main chapters. Here we introduce the thesis with an emphasis on its key components, providing a statement of problems under study. In the following we will make a brief presentation of each chapter.

Chapter 1 is based on the paper *Perturbation effects for a singular elliptic problem with lack of compactness and critical exponent* accepted in *Minimax Theory and its Applications*. In this chapter we study the problem

$$-\triangle u = V(x)|x|^{\alpha}|u|^{\frac{N+2+2\alpha}{N-2}} + \lambda g(x) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $\alpha \in (-2,0)$, $\lambda > 0$ is a real number, g belongs to an appropriate weighted Sobolev space, and V is a positive potential on $\mathbb{R}^N$.

The function V is supposed to satisfy the following hypotheses:

(V1) $V \in L^\infty(\mathbb{R}^N)$;

(V2) $\operatorname*{ess\,lim}_{|x|\to 0} V(x) = \operatorname*{ess\,lim}_{|x|\to\infty} V(x) = V_0 \in (0,\infty)$ and $V(x) \geq V_0$ a.a. $x \in \mathbb{R}^N$;

(V3) $\text{meas}(\{x \in \mathbb{R}^N : V(x) > V_0\}) > 0$.

The starting point of the variational approach of problem (1) is an inequality due to Caffarelli, Kohn and Nirenberg, which is based on Sobolev's and Hardy's inequalities, concretised in the next lemma.

Lemma 1. *Let $N \geq 2$, $\alpha \in (0,2)$ and denote $2^*_\alpha = \frac{2N}{N-2+\alpha}$. Then there is $C_\alpha > 0$ so that*

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha}\, dx\right)^{2/2^*_\alpha} \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2\, dx \tag{2}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$.

More specifically, we employ the suitable inequality

$$\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}}\, dx\right)^{\frac{N-2}{N+\alpha}} \leq C_\alpha \int_{\mathbb{R}^N} |\nabla u|^2\, dx \tag{3}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$, where $-2 < \alpha < 0$.

In this chapter we look for weak solutions of equation (1) in the Hilbert space $H^1(\mathbb{R}^N)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{1/2}.$$

We say that $u \in H^1(\mathbb{R}^N)$ is a weak solution of problem (1) if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx - \int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{N+2+2\alpha}{N-2}} v\, dx - \lambda \int_{\mathbb{R}^N} g(x) v\, dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}^N)$. We consider $g \in H^{-1}(\mathbb{R}^N)$, the dual space of $H^1(\mathbb{R}^N)$.

Also, we bring to light that the weak solutions of equation (1) correspond to the critical points of the energy functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} V(x)|x|^\alpha |u|^{\frac{2(N+\alpha)}{N-2}} dx - \lambda \int_{\mathbb{R}^N} g(x) u\, dx.$$

Given this background, the goal is to establish the existence of at least two weak solutions of equation (1). More exactly, we want to prove the existence of some $\lambda_* > 0$ such that our equation has two different solutions provided that $\lambda \in (0, \lambda_*)$. The arguments used rely essentially on Ekeland's variational principle, the mountain pass theorem without the Palais–Smale condition, a weighted version of Brezis–Lieb lemma, and the Caffarelli–Kohn–Nirenberg inequality.

Chapter 2 is based on the paper *On the existence of infinitely many solutions of a nonlinear Neumann problem involving the m-Laplace operator* submitted in *Annals of the University of Craiova, Mathematics and Computer Science Series.* In this chapter we study the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\partial/\partial\nu$ denotes the outward normal derivative, $f(x,u)$ and $g(x,u)$ are continuous functions on $\overline{\Omega} \times \mathbb{R}$ and on $\partial\Omega \times \mathbb{R}$, respectively, and odd with respect to u. A typical example is the next equation:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = a(x)|u|^{p-1}u & \text{in } \Omega, \\ |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = b(x)|u|^{q-1}u & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We assume that $a \in C(\overline{\Omega})$, $b \in C(\partial\Omega)$, $a(x)$ and $b(x)$ may change their signs, $a(x_1) > 0$ at some $x_1 \in \Omega$, $b(x_2) > 0$ at some $x_2 \in \partial\Omega$ and p, q satisfy either (6) or (7):

$$0 < q < m-1 < p < \frac{(m-1)N+m}{N-m}, \tag{6}$$

$$0 < p < m-1 < q < \frac{(m-1)N}{N-m}. \tag{7}$$

When $N = 1, 2, ..., m$, the right hand sides of (6) and (7) are replaced by ∞. We show that problem (5) has at least two sequences u_k and v_k of solutions such that

$$\|u_k\|_{W^{1,m}(\Omega)} \to 0, \quad \|u_k\|_{C(\overline{\Omega})} \to 0 \text{ as } k \to \infty,$$

$$\|v_k\|_{W^{1,m}(\Omega)} \to \infty, \quad \|v_k\|_{C(\overline{\Omega})} \to \infty \text{ as } k \to \infty.$$

Here $W^{1,m}(\Omega)$ is an usual Sobolev space, $\|\cdot\|_{W^{1,m}(\Omega)}$ denotes the $W^{1,m}(\Omega)$ norm and $\|\cdot\|_{C(\overline{\Omega})}$ is the maximum norm.

In Chapter 2 we shall introduce a *locally superlinear condition* and a *locally sublinear condition*. We prove the existence of infinitely many solutions. One of our goals is to prove that locally superlinear condition yields a sequence of solutions diverging to infinity and the locally sublinear condition yields a sequence of solutions converging to zero. Another purpose is to study the existence of at least two sequences of solutions such that one sequence converges to zero and another diverges to infinity in each of the following cases:

(i) one of $f(x,u)$ and $g(x,u)$ is locally sublinear and another is locally superlinear;

(ii) $f(x,u)$ is both locally sublinear and locally superlinear;

(iii) $g(x,u)$ is both locally sublinear and locally superlinear.

Chapter 2 is organized into five sections. In the second section of this chapter we give the main results. In the third section we present some examples of the nonlinear terms $f(x,u)$ and $g(x,u)$ and apply our theorems to them to show the existence of at least two sequences of solutions. In the fourth section we prove that a weak solution in $W^{1,m}(\Omega)$ belongs to $W^{1,r}(\Omega)$ for all $r<\infty$, and give $W^{1,r}(\Omega)$ a priori estimates. The last section contains the proofs of the main theorems via the variational method with the help of a priori $W^{1,r}(\Omega)$ estimates.

Chapter 3 is based on the paper *Existence and multiplicity of solutions for a class of isotropic elliptic equations with variable exponent* published in *Annals of the University of Craiova, Mathematics and Computer Science Series*. In this chapter we are concerned with the nonhomogeneous eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda|u|^{q(x)-2}u-h(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, $\lambda>0$ is a real number, and p, q are continuous on $\overline{\Omega}$ satisfying the basic assumption

$$2\leq p(x)<q(x)<r(x)<p^*(x),$$

where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\overline{\Omega}$, while $h:\overline{\Omega}\to[0,\infty)$ is a continuous function such that

$$\left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}}\in L^1(\Omega),$$

$$\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}} \in L^{\frac{r(\cdot)}{r(\cdot)-2}}(\Omega).$$

The goal of this chapter is to show the existence and multiplicity of solutions for problem (8). We shall work on a subspace of the *variable exponent Sobolev space* $W_0^{1,p(\cdot)}(\Omega)$ defined as

$$E = \left\{u \in W_0^{1,p(\cdot)}(\Omega);\ \int_\Omega h(x)|u|^{r(x)}dx < \infty\right\}.$$

The main results in the Chapter 3 point out the following perturbation effects:

(i) the first theorem states that if the perturbation in the right-hand side of (8) is strong (this corresponding to λ is sufficiently big), then there exist at least two different nontrivial solutions;

(ii) the second theorem states that if the perturbation in the right-hand side of (8) is weak (this corresponding to λ is sufficiently small), then there is no nontrivial solution.

We remember that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (8) if there exists $u \in W_0^{1,p(\cdot)}(\Omega)\backslash\{0\}$ satisfying $\int_\Omega h(x)|u|^{r(x)}dx < \infty$ such that

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right)dx - \lambda\int_\Omega |u|^{q(x)-2}uv\,dx + \int_\Omega h(x)|u|^{r(x)-2}uv\ dx = 0$$

for all $v \in E$. We point out that if λ is an eigenvalue of problem (8) then the corresponding $u \in E$ is a weak solution of (8).

The proof of the first theorem is divided into two steps. At *Step 1* we prove the existence of a nontrivial solution for problem (8). First, we show that the *energetic functional* $\Phi : E \to \mathbb{R}$ (whose critical points are exactly the weak solutions of (8)) defined by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx$$

is *coercive* and *weakly lower semicontinuous* on E and thus we deduce that there exists a weak solution $\tilde{u}$ of problem (8). Second, we shall demonstrate that $\tilde{u} \not\equiv 0$ in E. Next, at *Step 2*, the second weak solution $\hat{u}$ is obtained essentially using the mountain pass theorem and Sobolev embeddings, proving in the same tame that $\tilde{u} \neq \hat{u}$.

To prove the second theorem, that is, to establish the non-existence of nontrivial solutions for problem (8) we argue by contradiction.

Chapter 4 is based on the paper *Combined concave-convex effects in anisotropic elliptic equations with variable exponent* published in *Nonlinear Differential Equations and Applications NoDEA*. This chapter is devoted to the study of weak solutions for nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right) = \lambda|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real number, p_i, q, r are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $2 < q(x) < r(x)$ for any $x \in \overline{\Omega}$ and $i \in \{1, ..., N\}$, and $h : \overline{\Omega} \to \mathbb{R}$ is a continuous positive weight function satisfying the conditions:

$$\int_\Omega \lambda^{\frac{r(x)}{r(x)-q(x)}} \frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx < \infty,$$

$$\int_\Omega \left(\frac{\lambda}{h(x)^{\frac{q(x)-2}{r(x)-2}}}\right)^{\frac{r(x)}{r(x)-q(x)}} dx < \infty.$$

In Chapter 4 we seek solutions for problem (9) in a subspace E of the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ defined by

$$E = \left\{u \in W_0^{1,\vec{p}(\cdot)}(\Omega);\ \int_\Omega h(x)|u|^{r(x)}dx < \infty\right\}.$$

We say that $u \in E$ is a weak solution of equation (9) if $u = 0$ almost everywhere on $\partial\Omega$ and

$$\int_\Omega \left\{ \sum_{i=1}^{N} |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v - \lambda|u|^{q(x)-2}uv + h(x)|u|^{r(x)-2}uv \right\}dx = 0$$

for all $v \in E$.

The first main result of this chapter asserts the non-existence of nontrivial weak solutions for problem (9) if λ is small enough. To prove this first theorem we argue by contradiction.

Then, we intend to get the existence of at least two nontrivial weak solutions for problem (9) if λ is sufficiently large. In what concerns the proof of this second theorem it employs critical point theory. More exactly, we define the energy functional $\Phi : E \to \mathbb{R}$ defined by

$$\Phi(u) = \int_\Omega \left\{ \sum_{i=1}^{N} \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} - \frac{\lambda}{q(x)}|u|^{q(x)} + \frac{h(x)}{r(x)}|u|^{r(x)} \right\}dx$$

whose derivative is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \left\{ \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v - \lambda |u|^{q(x)-2}uv + h(x)|u|^{r(x)-2}uv \right\} dx$$

for all $u, v \in E$. Thus, the critical points of Φ are exactly the weak solutions of equation (9). In a first instance we show that there exists $u_1 \in E$ a global minimizer of Φ such that $\inf_E \Phi < 0$ and thus, we obtain a first non-trivial weak solution of problem (9).

Next, we set

$$g(x,t) = \begin{cases} 0, & \text{for } t < 0 \\ \lambda t^{q(x)-1} - h(x)t^{r(x)-1}, & \text{for } 0 \le t \le u_1(x) \\ \lambda u_1(x)^{q(x)-1} - h(x)u_1(x)^{r(x)-1}, & \text{for } t > u_1(x) \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s)\, ds.$$

Define the functional $\Psi : E \to \mathbb{R}$ by

$$\Psi(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \int_\Omega G(x,u) dx.$$

Obviously, $\Psi \in C^1(E, \mathbb{R})$ and

$$\langle \Psi'(u), v\rangle = \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v \, dx - \int_\Omega g(x,u)v\, dx,$$

for all $u, v \in E$.

We will find a critical point $u_2 \in E$ of Ψ such that $\Psi(u_2) > 0$ via the mountain pass theorem and we will show that $0 \le u_2 \le u_1$ in Ω. Therefore

$$g(x,u_2) = \lambda u_2^{q(x)-1} - h(x)u_2^{r(x)-1} \quad \text{and} \quad G(x,u_2) = \frac{\lambda}{q(x)} u_2^{q(x)} - \frac{h(x)}{r(x)} u_2^{r(x)}$$

and thus

$$\Psi(u_2) = \Phi(u_2) \quad \text{and} \quad \Psi'(u_2) = \Phi'(u_2).$$

More exactly we get

$$\Phi(u_2) > 0 = \Phi(0) > \Phi(u_1) \quad \text{and} \quad \Phi'(u_2) = 0.$$

This shows that u_2 is a second weak solution of problem (9) such that $0 \leq u_2 \leq u_1$, $u_2 \neq 0$ and $u_2 \neq u_1$.

Chapter 5 is based on the paper *Spectrum for anisotropic equations involving weights and variable exponents* published in *Electronic Journal of Differential Equations.* This chapter deals with the spectrum of a nonhomogeneous anisotropic problem involving variable exponents on a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 3$), that is

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \left[\partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) + |u|^{p_i(x)-2}u\right] + |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where p_i, q, $r : \overline{\Omega} \to [2, \infty)$ are Lipschitz continuous functions, while $g : \overline{\Omega} \to [0, \infty)$ is a measurable function for which there exists an open subset $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \geq 0$ is a real number.

In Chapter 5 we proposed to treat the problem (10) assuming that the functions p_i, q and r satisfy the hypotheses

$$2 \leq P_-^- \leq P_+^+ < N, \tag{11}$$

$$P_-^+ \leq P_+^+ < r^- \leq r^+ < q^- \leq q^+ < P_-^* \leq p_i^*(x) \quad \forall x \in \overline{\Omega} \text{ and } \forall i \in \{1, \ldots, N\}. \tag{12}$$

Moreover, we assume that the function $g(x)$ satisfies the hypotheses

$$\int_\Omega (\lambda g(x))^{\frac{q(x)}{q(x)-r(x)}}\, dx < \infty, \tag{13}$$

$$g \in L^\infty(\Omega) \cap L^{p_i^0(\cdot)}(\Omega), \tag{14}$$

where $p_i^0(x) = \frac{p_i^*(x)}{p_i^*(x) - r^-}$ for every $x \in \overline{\Omega}$ and every $i \in \{1, \ldots, N\}$.

We look for weak solutions for problem (10) in the *anisotropic variable exponent Sobolev space* $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (10) if there exists an element $u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \left[\sum_{i=1}^{N} \left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v + |u|^{p_i(x)-2}uv\right) + |u|^{q(x)-2}uv\right] dx - \lambda \int_\Omega g(x)|u|^{r(x)-2}uv\, dx = 0,$$

for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. If λ is an eigenvalue of the problem (10) then the corresponding $u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ is a *weak solution* of (10).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N \left(\frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_i(x)}}{p_i(x)} \right) dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx}{\displaystyle\int_\Omega \frac{g(x)}{r(x)} |u|^{r(x)} dx},$$

$$\lambda_0 := \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N \left(|\partial_{x_i} u|^{p_i(x)} + |u|^{p_i(x)} \right) dx + \int_\Omega |u|^{q(x)} dx}{\displaystyle\int_\Omega g(x) |u|^{r(x)} dx}.$$

The main result of this chapter asserts that under conditions (11)-(14) we have $0 < \lambda_0 \leq \lambda_1$ and any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (10), but every $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (10). With an eye to prove this result we shall define the functionals $J_1,\ I_1,\ J_0,\ I_0 : W_0^{1,\vec{p}(\cdot)}(\Omega) \to \mathbb{R}$ by

$$J_1(u) = \int_\Omega \sum_{i=1}^N \left(\frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_i(x)}}{p_i(x)} \right) dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx,$$

$$I_1(u) = \int_\Omega \frac{g(x)}{r(x)} |u|^{r(x)} dx,$$

$$J_0(u) = \int_\Omega \sum_{i=1}^N \left(|\partial_{x_i} u|^{p_i(x)} + |u|^{p_i(x)} \right) dx + \int_\Omega |u|^{q(x)} dx,$$

$$I_0(u) = \int_\Omega g(x) |u|^{r(x)} dx.$$

Standard arguments imply that $J_1,\ I_1 \in C^1(E, \mathbb{R})$ with the derivatives given by

$$\langle J_1'(u), v \rangle = \int_\Omega \left[\sum_{i=1}^N \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v + |u|^{p_i(x)-2} uv \right) + |u|^{q(x)-2} uv \right] dx,$$

$$\langle I_1'(u), v \rangle = \int_\Omega g(x) |u|^{r(x)-2} uv \, dx.$$

For every $\lambda > 0$ we also define the functional $T_\lambda^1(u) = J_1(u) - \lambda \cdot I_1(u)$ for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. We mention that λ is an eigenvalue of problem (10) if and only if there exists $u_\lambda \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$, which is a critical point of the functional T_λ^1.

We split the proof of our theorem into four steps. To Step 1 we show that $\lambda_0,\ \lambda_1 > 0$. To Step 2 we prove by contradiction that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (10). Then, to Step 3 we show that each $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (10). To this aim, we prove that the functional T^1_λ is coercive and weakly lower semicontinuous in order to obtain $u_\lambda \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, a global minimum point of T^1_λ and thus a critical point of T^1_λ. To complete the proof of Step 3 we also show that u_λ is not trivial. Finally, to Step 4 we demonstrate that λ_1 is an eigenvalue of problem (10). By Steps 2-4 we deduce that $\lambda_0 \leq \lambda_1$ completing the proof of the main result.

Chapter 6 is based on the paper *Eigenvalue problems for anisotropic equations involving a potential on Orlicz-Sobolev type spaces* published in *Opuscula Mathematica*. Here we study the anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i(\varphi_i(\partial_i u)) + V(x)|u|^{m(x)-2}u = \lambda(|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, $V : \Omega \to \mathbb{R}$ is a potential satisfying $V \in L^{r(x)}(\Omega)$, $r(x) \in C(\overline{\Omega})$, and q_1, q_2, $m : \overline{\Omega} \to (2, \infty)$ are continuous functions. Consider that, for each $i \in \{1, ..., N\}$, φ_i are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$.

In this chapter we look for weak solutions of problem (15) in a subspace of the *anisotropic Orlicz-Sobolev space* $W_0^1 L_{\vec{\Phi}}(\Omega)$, that is

$$E := \left\{ u \in W_0^1 L_{\vec{\Phi}}(\Omega);\ \int_\Omega |V(x)||u|^{m(x)} dx < \kappa, \text{ with } \kappa > 0 \text{ real constant} \right\}.$$

Define the functionals J_V, $I : E \to \mathbb{R}$ by

$$J_V(u) = \int_\Omega \sum_{i=1}^{N} \Phi_i(|\partial_i u|)dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx,$$
$$I(u) = \int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx.$$

Then, for each $\lambda \in \mathbb{R}$, we define the energetic functional associated with problem (15), $T_\lambda : E \to \mathbb{R}$, by

$$T_\lambda(u) = J_\lambda(u) - \lambda \cdot I(u).$$

We notice that $J_V,\ I \in C^1(E,\mathbb{R})$ with the derivatives given by

$$\begin{aligned}\langle J_V'(u),v\rangle &= \int_\Omega \sum_{i=1}^N a_i(|\partial_i u|)\partial_i u \partial_i v\ dx + \int_\Omega V(x)|u|^{m(x)-2}uv\ dx,\\ \langle I'(u),v\rangle &= \int_\Omega |u|^{q_1(x)-2}uv\ dx + \int_\Omega |u|^{q_2(x)-2}uv\ dx,\end{aligned}$$

for all $u,v\in E$. Consequently, $T_\lambda \in C^1(E,\mathbb{R})$ and

$$\langle T_\lambda'(u),v\rangle = \langle J_V'(u),v\rangle - \lambda\langle I'(u),v\rangle, \quad \forall\, u,v\in E.$$

Thus, λ is an eigenvalue of problem (15) if and only if there exists $u\in E\setminus\{0\}$, a critical point of T_λ.

In Chapter 6 we intend to prove three main results, among other auxiliary results.

The first theorem of this chapter asserts that any $\lambda>0$ is an eigenvalue of problem (15) under assumption

$$2 < (P^0)_+ < q_2^- \le q_2^+ \le m^- \le m^+ \le q_1^- \le q_1^+ < q_1^+ \cdot r^{-'} < (P_0)^*.$$

In what concerns the proof we use the mountain-pass lemma to obtain the existence of a sequence $(u_n)\subset E$ so as

$$T_\lambda(u_n) \to \bar{c} > 0 \text{ and } T_\lambda'(u_n) \to 0 \text{ (in } E^*) \text{ as } n\to\infty. \tag{16}$$

Also, we argue by contradiction that (u_n) is bounded in E. This information, combined with the fact that E is reflexive, implies that there exists a subsequence, still denoted by (u_n), and $u_0\in E$ such that (u_n) converges weakly to $u_0\in E$. This will lead us to

$$\sum_{i=1}^N \int_\Omega \Phi_i\left(\left|\frac{\partial_i u_n - \partial_i u_0}{2}\right|\right) dx \underset{n\to\infty}{\longrightarrow} 0, \tag{17}$$

which means that $u_n\to u_0$ in E. Thus, (16) and (17) yield

$$T_\lambda(u_0) = \bar{c} > 0 \ \text{ and } \ T_\lambda'(u_0) = 0,$$

meaning that $u_0\in E$ is a nontrivial weak solution of problem (15).

The second theorem of Chapter 6 asserts that under assumption

$$2 < q_2^- \le q_2^+ \le q_1^- \le q_1^+ \le m^- \le m^+ < m^+ \cdot r^{-'} < (P_0)_- \le (P_0)^*,$$

there exists $\lambda_* > 0$ such that any $\lambda \in (0, \lambda_*]$ is an eigenvalue of problem (15).

First, we prove that there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*]$ there exist $\rho > 0$ and $a > 0$ so as $T_\lambda(u) \geq a > 0$ for every $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$. That is to say, on the ball $B_\rho(0)$ we have $\inf_{\partial B_\rho(0)} T_\lambda > 0$. Second, we show that there exists $\theta \in E$ such that $\theta \geq 0$, $\theta \not\equiv 0$ and $T_\lambda(t\theta) < 0$ for $t > 0$ small enough. In fact, we can prove that $-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} T_\lambda < 0$. Relying on variational arguments based on Ekeland's variational principle, we deduce that there exists a sequence $(w_n) \subset B_\rho(0)$ such that

$$T_\lambda(w_n) \to \underline{c} \ \text{ and } \ T'_\lambda(w_n) \to 0. \tag{18}$$

It is clear that (w_n) is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, (w_n) converges weakly to w in E. Actually, we get that (w_n) converges strongly to w in E. By (18) we finally obtain

$$T_\lambda(w) = \underline{c} < 0 \ \text{ and } \ T'_\lambda(w) = 0,$$

i.e., w is a nontrivial weak solution for problem (15).

The third theorem of Chapter 6 asserts that under assumption

$$2 < q_2^- \leq q_2^+ \leq m^- \leq m^+ \leq q_1^- \leq q_1^+ < q_1^+ \cdot r^{-\prime} < (P_0)_- \leq (P_0)^*,$$

there exists $\lambda^* > 0$ such that any $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of problem (15).

To prove this last theorem we keep in mind that the functional T_λ is coercive and weakly lower semicontinuous on E. Thereby, we obtain an element $\underline{u} \in E$, global minimizer of T_λ and therefore a weak solution of problem (15). Furthermore, we show that $\underline{u}$ is not trivial for λ large enough.

The thesis closes with a list of references which includes 137 works.