University of Craiova
Faculty of Sciences
Doctoral School of Sciences

Luisa-Maria Niţu

Classes of filters in algebras of fuzzy logic

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supervisor: Prof. univ. dr. Dumitru Buşneag

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Contents

Chapter 1

Introduction

The aim of this thesis is to extend some results from lattices to the case of residuated lattices. In this thesis we work with commutative residuated lattices and non-commutative residuated lattices. In the rest of the thesis we understand through a residuated lattice, a commutative residuated lattice and through a bounded integral residuated lattice, a non-commutative residuated lattice. We realize a study on distributivity of residuated lattices and we give a new characterization for boolean, regular, stong and dense elements in a bounded integral residuated lattice. Also, we extend some results from normal lattice to the case of i-normal residuated lattice defined in [17], [18] and starting from [22], [23] we realize a classification of implicative filters in residuated lattices. The subject of this thesis it is situated in the field of the algebra of logic. The results we are presenting in this thesis can be found in the following three papers belonging to the author of this thesis: [29], [30] and [65]. In this introduction, we present our main motivation for studying these topics and give an overview of the thesis.

1.1 Multiple-valued logics and their algebras

Between mathematical logic and algebra there is a complex relationship and are closely related. The algebra of logic and algebraic logic are two of the instances representing this relationship (see [47]). Mathematical logic is a discipline that belongs Mathematics and Logic equally.

Algebraic methods were introduced in logic by George Boole in four writings ([10], [11], [12] and [13]) published around the year 1850. The term "algebra of logic" was imposed by Boole's successors (De Morgan, Schröder and Pierce). The first work on the logic of Moisil it is called "Recherches sur l'algèbre de la logique" (see [94]). The notion of "Boolean algebra" appears in [124]. The theory of Boolean algebras was extensively developed by

Stone, Birkhoff and Tarski in the four-decade of the last century.

M. Ward and R. P. Dilworth were the first who introduced the important concept of a *residuated lattice* ([122],[123]) as a generalization of ideal lattices of rings. In the definition they are using, a residuated lattice is what we would call it an *integral commutative* one. The general definition of a residuated lattice, as used today, was given by K. Blount, P. Jipsen, T. Kowalski and H. Ono in [45].

Residuated lattices include important classes of algebras such as BL-algebras (defined by P. H*á*jek as the algebraic counterpart of his Basic Logic) and MV-algebras (defined by Chang in [31] to prove the completeness theorem for Łukasiewicz calculus). In 1998, P. H*á*jek ([61]) introduced the notion of BL-algebras, also, defined the concepts of filters and prime filters in BL-algebras. Using prime filters in BL-algebras, he managed to prove the completeness of Basic Logic. In 1999, E. Turunen ([114]) published a study on BL-algebras and their deductive systems.

Residuated lattices have important algebraic properties (see [27], [61], [108] and [114]).

In [27], it is presented a new characterization of complemented elements in a residuated lattice, that uses the concepts of *regular* and *dense elements*.

P. M. Idziak proved in [68], that the class of residuated lattices is equational. Over time, these lattices have been known under many names: *BCK- latices* in [64], *full BCK- algebras* in [82], FL_{ew}*- algebras* in [102], also, *integral, residuated, commutative l-monoids* in [4], [5].

Non-commutative residuated lattices, in some cases called *pseudo-residuated lattices*, are the algebraic counterparts of sub-structural logics. Studies on non-commutative residuated lattices were developed by Ono, Jipsen, Galatos, Tsinakis and Kowalski in [45] and [81]. Particular classes of non-commutative residuated lattices are full Lambek algebras (*FL-algebras)* and *bounded integral residuated lattices* (FL_w*- algebras*).

In 1999, G. Georgescu and A. Iorgulescu ([50]) defined *the pseudo-BL algebras* as a generalization of BL-algebras for the general non-commutative case. Properties of pseudo-BL algebras were intensely developed by A. Di Nola, G. Georgescu and A. Iorgulescu in [40] and [41]. Classes of pseudo-BL algebras were investigated in [53] and the corresponding propositional logic was established and thorough by Hájek in [61] and [62].

A more general structure than the pseudo-BL algebra is the case of *pseudo-MTL algebra* (or *weak pseudo-BL algebra*) defined by P. Flondor, G. Georgescu and A. Iorgulescu in [44]. The properties and characteristics of pseudo-MTL algebras are intensively studied in [35], [54], [74] and [75].

In 2014, L. C. Ciungu published ([36]), an important study in the field of non-commutative multiple-valued logic algebras.

1.2 Overview of the thesis

The thesis is organized as follows: **Chapter 2** is dedicated to basic definitions and results about the structures used in the thesis.

In **Section 1** and **2** we recall the basic definitions, we put in evidence many rules of calculus, classes of residuated lattices and examples of residuated lattices which we need in the rest of the thesis.

In **Section 3** we put in evidence some sufficient conditions for distributivity of residuated lattices (see Theorem 2.5). We managed to get a partial characterization of distributive residuated lattices just with sufficient conditions

We recall some classes of residuated lattices and we realize a detailed classification on the basis of their distributivity.

Section 4 contains definitions and known results about boolean center , regular and dense elements in residuated lattices that we extend in chapter 3 in the case of non-commutative residuated lattices.

In **Section 5** we present a study on implicative filters (i-filter, for short) theory in residuated lattices. We mention some examples to see slightly different properties. We focus our study on properties of implicative filters in residuated lattices.

In **Section 6** we recall some notions relative to morphisms of residuated lattices and direct product of residuated lattices and we present examples and properties.

In **Chapter 3**, we work in the general setting of a non-commutative residuated lattice (more precisely, we work with *bounded integral residuated lattices*, or FL_w- *algebras,* for short) and extend to the non-commutative case some results from [27].

Section 1 containes basic definitions, examples, many rules of calculus and classes of bounded integral residuated lattices.

Using [36] (Theorem 4.4 and Propositions 2.1, 2.2) we establish a connection between a bounded integral residuated lattice and a pseudo MV-algebra:

In the end of this section we present two properties in a bounded integral residuated lattice which satisfies

(W) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$ and $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

In **Section 2** an important construction for a bounded integral residuated lattice A is the boolean center $B(A)$, which contains all complemented elements of A. We recall the definitions and properties of Boolean center and boolean elements.

We give a new characterization for involutive and idempotent elements in a bounded integral residuated lattice.

In this section we put in evidence new rules of calculus with boolean elements in a bounded integral residuated lattice A (see Lemma 3.4).

We introduce definitions of regular, strong and dense elements in a bounded integral residuated lattice and we give new characterizations for this elements.

In **Chapter 4**, we transfer some results from normal lattice to the case of i-normal residuated lattice defined in [17], [18].

Section 1 contains definitions and properties known about normal lattices.

Definition 4.1. A bounded distributive lattice L which verifies the condition: for any $x, y \in L$ with $x \wedge y = 0$, there exist $z, t \in L$ such that $x \wedge z = y \wedge t = 0$ and $z \vee t = 1$, is called *normal*.

L is called *co-normal* iff it is dually normal (that is, for all $x, y \in L$, if $x \vee y = 1$, then there exist $z, t \in L$ such that $z \wedge t = 0$ and $z \vee x = t \vee y = 1$).

In **Section 2**, we transfer definitions and some results from normal lattice to the case of i-normal residuated lattice and we make a characterization of i-normal residuated lattices with the help of the co-normal lattice $\mathcal{F}_{ip}(L)$.

In [48], the i-normal residuated lattices are called *residuated lattices with the Gelfand property* (or *Gelfand residuated lattices*). Gelfand's property holds in Stonean residuated lattices, BL-algebras, but does not hold in any residuated lattice.

In this section, we introduced the notion of comaximal i-filter in a residuated lattice and the notion annihilator of a relative to b.

Two i-filters P and Q in a residuated lattice L are called *comaximal* iff $P \vee_i Q = L$.

If L is a residuated lattice and $a, b \in L$, then we consider the set
$\langle a, b\rangle = \{x \in L : a^n \odot x \leq b$ for some $n \geq 1\}$ (is called the annihilator of a relative to b ([91])).

We demonstrate that the annihilator of a relativ to b in a residuated latice is an ideal and we realize a characterization of i-normal residuated lattices.

In **Chapter 5**, we realize a study of implicative filters in residuated lattices and we establishe a important classification and connections between these types of filters.

In [22], it is proposed that the main names of implicative filters F of a residuated lattice L to be represented by the name of the class of algebras which contains the residuated lattice L/F.

Let $\mathcal{V}$ be a subvariety of the variety of residuated lattices.

We realize a classification of implicative filters in residuated lattices as follows:

– the class $\mathcal{V} =$ **BF** of Boolean i-filters;
– the class $\mathcal{V} =$ **GF** of Heyting i-filters;
– the class $\mathcal{V} =$ **MVF** of MV i-filters;
– the class $\mathcal{V} =$ **RF** of involution i-filters;
– the class $\mathcal{V} =$ **SgF** of semi-G i-filters;
– the class $\mathcal{V} =$ **StF** of Stonean i-filters;
– the class $\mathcal{V} =$ **MTLF** of MTL i-filters;
– the class $\mathcal{V} =$ **DivF** of divisible i-filters;
– the class $\mathcal{V} =$ **BLF** of BL i-filters.

Chapter 2

Preliminaries

2.1 Residuated Lattices. Basic Notions

Definition 2.1. ([*45*], [*123*]) An algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ will be called *commutative integral residuated lattice* (*residuated lattice*) if

Lr_1 : $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

Lr_2 : $(L, \odot, 1)$ is a commutative monoid;

Lr_3 : for every $x, y, z \in L$, $x \leq y \to z$ iff $x \odot y \leq z$.

We denote by $\mathcal{RL}$ the class of residuated lattices.

2.1.1 Examples of residuated lattices

We put in evidence examples of residuated lattices ([76]).

2.1.2 Rules of calculus in residuated lattices

In what follows by L we denote the universe of a residuated lattice. For $x \in L$ we define $x^* = x \to 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for $n \geq 1$.

For $x, y, z \in L$, we have many rules of calculus ([27],[17],[45],[108],[114]).

2.2 Classes of residuated lattices

We recall some classes of residuated lattices and we realize a detailed classification on the basis of their distributivity.

2.3 Distributive residuated lattices

In this section we put in evidence some sufficient conditions for distributivity of residuated lattices (see Theorem 2.5). We managed to get a partial characterization of distributive residuated lattices just with sufficient conditions

Based on the distributive property we divide the residuated lattices in two classes: the distributive residuated lattices (G-algebras, MTL-algebras, BL-algebras, divisible residuated lattices, MV-algebras, involutive MTL-algebras, weak nilpotent minimum algebra, NM-algebras, respectively Product algebras) and the non-distributrive residuated lattices.

2.3.1 Sufficient conditions for distributivity in a residuated lattice

2.4 Boolean center, regular and dense elements in residuated lattices

This section contains definitions and known results about boolean center , regular and dense elements in residuated lattices that we extend in chapter 3 in the case of non-commutative residuated lattices.

2.5 Implicative filters in residuated lattices

In this section we present a study on implicative filters (i-filter, for short) theory in residuated lattices. We mention some examples to see slightly different properties. We focus our study on properties of implicative filters in residuated lattices.

2.5.1 Filters (ideals) in a lattice

In the beginning for easier understanding of i-filter theory in residuated lattices we recall some notions and properties of filters (ideals) in lattice theory.

2.5.2 Implicative filters in residuated lattices

We denote by $\mathcal{F}_i(L)$ ($\mathcal{F}_{ip}(L),\ Spec_i(L),\ Max_i(L)$) the set of all i-filters (the set of all principal i-filters, the set of all proper prime i-filters and the set of all proper maximal i-filters) of L. Clearly, $\mathcal{F}_{ip}(L) \subseteq \mathcal{F}_i(L)$ and $Max_i(L) \subseteq Spec_i(L)$.

Following the Prime i-filter theorem we deduce an important corollary (see Corollary 2.6) and with the Principle of the minimal element we are showing a new theorem which is based of the minimal prime i-filter (see Theorem 2.12).

Corollary 2.6.

(i) Any proper i-filter F of L can be extended to a prime i-filter and F is the intersection of these prime i-filters which contain F;

(ii) If $I \in \mathcal{I}_d(L)$ and $I \neq L$ then there is a prime i-filter P of L such that $P \cap I = \varnothing$;

(iii) If $F \in \mathcal{F}_i(L)$ and $a \in L$ such that $a \notin F$, then there is a prime i-filter P such that $F \subseteq P$ and $a \notin P$;

(iv) If $a \in L, a \neq 1$, then there is a prime i-filter P of L such that $a \notin P$;

(v) $\bigcap \{P : P \in Spec_i(L)\} = \{1\}$.

Theorem 2.12. If $F \in \mathcal{F}_i(L)$ is proper and $S \subseteq L$ is a $\vee-$closed set such that $F \cap S = \varnothing$, then there is a minimal prime i-filter P belonging to F with the property that $P \cap S = \varnothing$.

For $P \in Spec_i(L)$ we consider $\mathbf{1}(P) = \{x \in L : x \vee y = 1 \text{ for some } y \notin P\}$. Clearly, $\mathbf{1}(P) \subseteq P$.

Lemma 2.4. $\mathbf{1}(P) \in \mathcal{F}_i(L)$.

Also, we put in evidence and three corollaries (Corollary 2.8, Corollary 2.9 and Corollary 2.10) of the Proposition 2.20 which is based of the minimal prime i-filter and, presented in [17].

We have the following corollaries of Proposition 2.20 :

Corollary 2.8. Let $F \in \mathcal{F}_i(L), n \in \mathbb{N}^*$ and $P_0, P_1, ..., P_n$ be $n+1$ distinct minimal prime i-filters belonging to F. Then there are $a_0, a_1, ..., a_n \in L$ such that $a_i \vee a_j \in F$ for all $i, j \in \overline{0, n}$ with $i \neq j$ and $a_i \notin P_i$ for all $i \in \overline{0, n}$.

Corollary 2.9. If $P \in Spec_i(L)$ is minimal, then for every $x \in P$, there is a $y \notin P$ such that $x \vee y = 1$.

Corollary 2.10. If $P \in Spec_i(L)$, then each minimal prime i-filter Q belonging to $\mathbf{1}(P)$ is included in P.

For $F \in \mathcal{F}_i(L)$ we define $\mathbf{0}(F) = \{x \in L : x^n \odot y = 0 \text{ for some } y \in F \text{ and } n \geq 1\}$.

Lemma 2.5. If $F \in \mathcal{F}_i(L)$, then $\mathbf{0}(F) \in \mathcal{I}_d(L)$.

Proposition 2.22. For $M \in \mathcal{F}_i(L), M \neq L$, the following assertions are equivalent

(i) $M \in Max_i(L)$;

(ii) $L \setminus M = \mathbf{0}(M)$.

In the end of this section we recall some classes of filters in residuated lattices that we use in classification from the chapter 5.

Let $F \in \mathcal{F}_i(L)$ and $x, y, z \in L$. In what follows, we enumerate some conditions which will be used in this thesis (see [14], [15], [21], [46], [63], [86], [90], [112], [116], [125] – [130]):

(F_4) for every $x \in L$, $x \vee x^* \in F$;

(F_5) If $x \to (z^* \to y), y \to z \in F$, then we have $x \to z \in F$;

(F_6) If $x \to (y \to z), x \to y \in F$, then we have $x \to z \in F$;

(F_7) If $(x \to y) \to x \in F$, then we have $x \in F$;

(F_8) for every $x \in L$, $x \to x^2 \in F$;

(F_9) If $x \to y \in F$, then we have $((y \to x) \to x) \to y \in F$;

(F_{10}) If $z^{**} \to (x \to y), z^{**} \to x \in F$, then we have $z^{**} \to y \in F$;

(F_{11}) for every $x \in L$, $x^{**} \to x \in F$;

(F_{12}) for every $x, y \in L$, $(x \to y) \vee (y \to x) \in F$;

(F_{13}) for every $x \in L$, $x \in F$ or $x^* \in F$.

As follows, we recall some names for some classes of filters in residuated lattice L.
Lemma 2.15. $F \in \mathcal{F}_i(L)$ is called:

(i) A *Boolean* filter if it satisfies the condition (F_4) (see [21], [63], [80], [130]);

(ii) An *implicative ds* if it satisfies the condition (F_5) (see [116]);

(iii) An *implicative filter* if it satisfies the condition (F_6) (see [21], [63], [80], [130]);

(iv) A *positive implicative filter* if it satisfies the condition (F_7) (see [21], [46], [80], [130]);

(v) A *Heyting filter* (or *G-filter*) if it satisfies the condition (F_8) (see [21], [46], [80], [130]);

(vi) A *fantastic filter* if it satisfies the condition (F_9) (see [14], [80]);

(vii) An *easy filter* if it satisfies the condition (F_{10}) (see [21]);

$(viii)$ An *involution filter* (see [46], p. 3014) or *regular filter* (see [130]) if it satisfies the condition (F_{11});

(ix) An *MTL filter* if it satisfies the condition (F_{12}) (see [129]);

(x) An *obstinate filter* if it satisfies the condition (F_{13}) (see [14]).

2.6 Morphisms of residuated lattices and direct product of residuated lattices

In this section we recall some notions relative to morphisms of residuated lattices and direct product of residuated lattices and we present examples and properties. A property presented is that direct products of i-filters from Definition 2.15 are filters of the same type (see Remark 2.25). We are showing the existence of a covariant functor B from the category of residuated lattices to the category of Boolean algebras and that B preserve injective morphisms and direct products.

Remark 2.25. 1.) If F_i are Boolean i-filters, then $F = \prod_{i \in I} F_i$ is Boolean.
2.) If F_i are implicative ds, then $F = \prod_{i \in I} F_i$ is implicative.
3.) If F_i are implicative i-filters, then $F = \prod_{i \in I} F_i$ is implicative filter.
4.) If F_i are positive implicative i-filters, then $F = \prod_{i \in I} F_i$ is positive implicative filter.
5.) If F_i are Heyting i-filters, then $F = \prod_{i \in I} F_i$ is Heyting filter.
6.) If F_i are fantastic i-filters, then $F = \prod_{i \in I} F_i$ is fantastic filter.
7.) If F_i are easy i-filters, then $F = \prod_{i \in I} F_i$ is easy filter.
8.) If F_i are involutive i-filters, then $F = \prod_{i \in I} F_i$ is involutive filter.
9.) If F_i are MTL i-filters, then $F = \prod_{i \in I} F_i$ is MTL filter.
10.) If F_i are obstinate i-filters, then $F = \prod_{i \in I} F_i$ is obstinate filter.

Remark 2.26. The assignaments $A \to B(A)$ and $f \to B(f)$ defines a covariant functor $B : \mathcal{RL} \to \mathcal{B}$, from the category of residuated lattices to the category of Boolean algebras.

Proposition 2.25. B preserves injective morphisms.

Proposition 2.26. B preserves direct products.

Remark 2.27. 1.) If F is a i-filter of A, then $F \cap B(A)$ is a filter of $B(A)$.
2.) If F is a Boolean i-filter of A, then $F \cap B(A)$ is a Boolean filter of $B(A)$.
3.) If F is an implicative ds of A, then $F \cap B(A)$ is an implicative filter of $B(A)$.
4.) If F is an implicative i-filter of A, then $F \cap B(A)$ is an implicative filter of $B(A)$.
5.) If F is a positive implicative i-filter of A, then $F \cap B(A)$ is a positive implicative filter of $B(A)$.
6.) If F is a Heyting i-filter of A, then $F \cap B(A)$ is a Heyting filter of $B(A)$.
7.) If F is a fantastic i-filter of A, then $F \cap B(A)$ is a fantastic filter of $B(A)$.
8.) If F is an easy i-filter of A, then $F \cap B(A)$ is an easy filter of $B(A)$.
9.) If F is an involution i-filter of A, then $F \cap B(A)$ is an involution filter of $B(A)$.
10.) If F is an MTL i-filter of A, then $F \cap B(A)$ is an MTL filter of $B(A)$.
11.) If F is an obstinate i-filter of A, then $F \cap B(A)$ is an obstinate filter of $B(A)$.

12.) If G is a filter of $B(A)$, then $\langle G\rangle$ is a i-filter of A.

Chapter 3

New characterizations for special elements in bounded integral residuated lattices

In this chapter, we work in the general setting of a non-commutative residuated lattice (more precisely, we work with *bounded integral residuated lattices*, or FL_w- *algebras,* for short) and extend to the non-commutative case some results from [27].

3.1 Bounded integral residuated lattices

This section containes basic definitions, examples, many rules of calculus and classes of bounded integral residuated lattices.

Using [36] (Theorem 4.4 and Propositions 2.1, 2.2) we establish a connection between a bounded integral residuated lattice and a pseudo MV-algebra:

Theorem 3.1. A bounded integral residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo MV-algebra iff it satisfies two additional conditions:

(W) $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$ and $(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$,

(H) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$, for any $x, y \in A$.

In the following proposition is a characterization of a bounded RL-monoid:

Proposition 3.3. For a bounded integral residuated lattice A the following conditions are equivalent:

(i) A is a bounded RL-monoid;

(ii) $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \odot [(x \wedge y) \rightsquigarrow z]$ and $x \rightarrow (y \wedge z) = [(x \wedge y) \rightarrow z] \odot (x \rightarrow y)$, for every $x, y, z \in A$.

In the following tow propositions we establish a connection between a Hilbert algebra and a pseudo MV-algebra, respectively a relative Stone lattice.

Proposition 3.5. For a bounded integral residuated lattice $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ the following are equivalent:

(i) $(A, \to, 1)$ is a Hilbert algebra or $(A, \rightsquigarrow, 1)$ is a Hilbert algebra;

(ii) $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a G-algebra.

Proposition 3.6. For a bounded integral residuated lattice $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ the following are equivalent:

(i) $(A, \to, 1)$ ($(A, \rightsquigarrow, 1)$) is a Hilbert algebra;

(ii) $(A, \vee, \wedge, \to, 0, 1)$ ($(A, \vee, \wedge, \rightsquigarrow, 0, 1)$) is a relative Stone lattice.

In the end of this section we present two properties in a bounded integral residuated lattice which satisfies (W).

Lemma 3.1. Let A be a bounded integral residuated lattice which satisfies (W). Then

(i) $x^{-\sim} = x^{\sim -} = x$ for every $x \in A$;

(ii) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x = (x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = x \vee y$ for every $x, y \in A$.

3.2 Boolean center, regular, strong and dense elements in a bounded integral residuated lattice

In this section an important construction for a bounded integral residuated lattice A is the boolean center $B(A)$, which contains all complemented elements of A. We recall the definitions and properties of Boolean center and boolean elements.

For any bounded integral residuated lattice A, let us denote by $Id(A) = \{x \in A : x \odot x = x\}$ the set of all *idempotent elements* of A and by $Inv(A) = \{x \in A : x = (x^{\sim})^{-} = (x^{-})^{\sim}\}$ the set of all *involutive elements* of A. $Inv(A)$ is called the *involutive center* of A.

We give a new characterization for involutive and idempotent elements in a bounded integral residuated lattice.

Proposition 3.9. Let A be a bounded integral residuated lattice and $a, x \in A$. Then the following assertions are equivalent:

(i) $x \in Inv(A)$;

(ii) $a \to x = x^{-} \rightsquigarrow a^{-}$ and $a \rightsquigarrow x = x^{\sim} \to a^{\sim}$, for every $a \in A$.

Lemma 3.3. Let A be a bounded integral residuated lattice and $x \in Id(A)$. Then $x \odot (x \rightsquigarrow y) = x \odot y$ and $(x \to y) \odot x = y \odot x$, for every $y \in A$.

Proposition 3.10. If A is an RL- monoid and $x \in Id(A)$, $y \in A$, then

(i) $x \odot y = x \wedge y = y \odot x$;

(ii) $x \wedge x^{\sim} = 0 = x \wedge x^{-}$;

(iii) $x \rightsquigarrow y = x \to y$;

(iv) $x^{\sim} = x^{-}$;

(v) $x \to x^{-} = x \to x^{\sim} = x \rightsquigarrow x^{-} = x \rightsquigarrow x^{\sim} = x^{-} = x^{\sim}$.

In this section we put in evidence new rules of calculus with boolean elements in a bounded integral residuated lattice A (see Lemma 3.4).

We introduce definitions of regular, strong and dense elements in a bounded integral residuated lattice and we give new characterizations for this elements.

Definition 3.3. Let A be a bounded integral residuated lattice.We say that an element $x \in A$ is *regular* if for every $y \in A$ we have $(x \to y) \rightsquigarrow x = (x \rightsquigarrow y) \to x = x$. We denote by $R(A)$ the set of all regular elements of A. We say that an element $x \in A$ is *dense* if for every $r \in R(A)$ we have $x \to r = x \rightsquigarrow r = r$. We denote by $D(A)$ the set of all dense elements of A.

We give new characterizations for regular elements:

Theorem 3.2. Let A be a bounded integral residuated lattice. For $x \in A$ the following assertions are equivalent:

(i) $x \in R(A)$;

(ii) $x^{-} \rightsquigarrow x = x^{\sim} \to x = x$;

(iii) $x = x^{-\sim} = x^{\sim-}$ and $x^{-} \odot (x^{-} \rightsquigarrow x) = (x^{\sim} \to x) \odot x^{\sim} = 0$.

We introduce the definition of the strong element because together with regular and idempotent elements help us to characterize boolean elements.

Definition 3.4. Let A be a bounded integral residuated lattice. Using the model of [76], we say that an element $x \in A$ is *strong* if $x^{-} = x^{\sim}$. We denote by $S(A)$ the set of all strong elements of A. Clearly, if A is a commutative bounded integral residuated lattice, then $S(A) = A$.

We give a new characterization for boolean elements:

Theorem 3.3. Let A be a bounded integral residuated lattice. For $x \in A$ the following assertions are equivalent:

(i) $x \in B(A)$;

(*ii*) $x \in R(A) \cap Id(A) \cap S(A)$ and $(x \rightsquigarrow x^-) \vee (x^- \rightsquigarrow x) = (x \to x^\sim) \vee (x^\sim \to x) = 1$.

Corollary 3.2. If A is a pseudo $MTL-$ algebra, then $B(A) = R(A) \cap Id(A) \cap S(A)$.

From Theorems 3.2 and 3.3 we deduce that:

Corollary 3.3. If A is a bounded integral residuated lattice, then $B(A) \subsetneqq R(A) \subsetneqq Inv(A)$.

We characterize the bounded integral residuated lattices which are Boolean algebras:

Theorem 3.4. For a bounded integral residuated lattice A the following assertions are equivalent :

(*i*) A is a Boolean algebra relative to the natural ordering;

(*ii*) A is a G - algebra and $x^{-\sim} = x^{\sim -} = x$, for every $x \in A$.

We give new characterizations for dense elements and put the existing connections with boolean, regular and strong elements.

Theorem 3.5. Let A be a bounded integral residuated lattice. For $x \in A$ the following assertions are equivalent:

(*i*) $x \in D(A)$;

(*ii*) $x^- = x^\sim = 0$;

(*iii*) $(a \odot x)^- = a^-$ and $(x \odot a)^\sim = a^\sim$, for every $a \in A$.

Remark 3.18.

1. $D(A) \subseteq S(A)$;
2. Using Theorem 3.2, $c_1^{'}$ and $c_{25}^{'}$ we deduce that $D(A) \cap B(A) = D(A) \cap R(A) = D(A) \cap Inv(A) = \{1\}$;
3. From Theorem 3.5, (*iii*), we deduce that if $x \in D(A)$, then $(x^n)^- = (x^n)^\sim = 0$, so, $x^n \in D(A)$, for every $n \geq 1$.

Chapter 4

Normal residuated lattices

In this chapter, we transfer some results from normal lattice to the case of i-normal residuated lattice defined in [17], [18].

4.1 Normal lattices

This section contains definitions and properties known about normal lattices.

Definition 4.1. A bounded distributive lattice L which verifies the condition: for any $x, y \in L$ with $x \wedge y = 0$, there exist $z, t \in L$ such that $x \wedge z = y \wedge t = 0$ and $z \vee t = 1$, is called *normal*.

L is called *co-normal* iff it is dually normal (that is, for all $x, y \in L$, if $x \vee y = 1$, then there exist $z, t \in L$ such that $z \wedge t = 0$ and $z \vee x = t \vee y = 1$).

4.2 i-Normal residuated lattices

In this section, we transfer definitions and some results from normal lattice to the case of i-normal residuated lattice and we make a characterization of i-normal residuated lattices with the help of the co-normal lattice $\mathcal{F}_{ip}(L)$.

Definition 4.2. A residuated lattice L is called *i-normal* if every prime i-filter P of L is contained in a unique maximal i-filter M_P.

Theorem 4.3. For a residuated lattice L, the following are equivalent:

(i) L is i-normal;

(ii) The lattice $\mathcal{F}_{ip}(L)$ is co-normal;

(iii) The lattice $\mathcal{F}_i(L)$ is co-normal.

In [48], the i-normal residuated lattices are called *residuated lattices with the Gelfand property* (or *Gelfand residuated lattices*). Gelfand's property holds in Stonean residuated lattices, BL-algebras, but does not hold in any residuated lattice.

Proposition 4.5. Consider L a Stonean residuated lattice. If L is a normal lattice, then L is an i-normal residuated lattice.

In this section, we introduced the notion of comaximal i-filter in a residuated lattice and the notion annihilator of a relative to b.

Two i-filters P and Q in a residuated lattice L are called *comaximal* iff $P \vee_i Q = L$.

Theorem 4.4. For a residuated lattice L the following assertions are equivalent:

(i) Every prime i-filter contains a unique minimal prime i-filter;

(ii) Any two distinct minimal i-filters are comaximal;

(iii) Every maximal i-filter contains a unique minimal i-filter.

If L is a residuated lattice and $a, b \in L$, then we consider the set $\langle a, b\rangle = \{x \in L : a^n \odot x \leq b \text{ for some } n \geq 1\}$ (is called the annihilator of a relative to b ([91])).

We demonstrate that the annihilator of a relativ to b in a residuated latice is an ideal and we realize a characterization of i-normal residuated lattices.

Lemma 4.1. For every $a, b \in L, \langle a, b\rangle \in \mathcal{I}_d(L)$.

For $I_1, I_2 \in \mathcal{I}_d(L)$, we denote by $I_1 \vee_{id} I_2$ the ideal of L generated by $I_1 \cup I_2$.

Clearly ([2]), $I_1 \vee_{id} I_2 = \{x \in L : x \leq a \vee b, \text{ with } a \in I_1 \text{ and } b \in I_2\}$.

Theorem 4.5. For a residuated lattice L, the following conditions are equivalent

(i) L is i-normal;

(ii) If $a, b \in L$ and $a \odot b = 0$, then $\langle a, b\rangle \vee_{id} \langle b, a\rangle = L$;

(iii) For every $P \in Spec_i(L)$ and $a, b \in L$ such that $a \odot b = 0$, there are $x \in P$ and $p, q, r \geq 1$ such that $a^p \odot x^q$ and $b \odot x^r$ are comparable.

Chapter 5

A new approach for classification of filters in residuated lattices

Main results underlying filters classification are presented for each section:

5.1 Classes of filters in residuated lattices

Definition 5.1 A filter $F \in \mathcal{F}_i(L)$ will be called a $\mathcal{V}-$ *filter* (or *filter of type* $\mathcal{V}$) if $L/F \in \mathcal{V}$.

5.2 The class $\mathcal{V} = \mathbf{B}$ (the subvariety of Boolean algebras)

Theorem 5.1 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:
(i) $F \in F(\mathbf{B})$;
(ii) F satisfies the condition (F_4);
(iii) F satisfies the condition (F_5);
(iv) F satisfies the condition (F_7).

5.3 The class $\mathcal{V} = \mathbf{BF}$ (of Boolean filters in a residuated lattice)

Proposition 5.2 For a i-filter F of L, the following conditions are equivalent:
(i) F is a Boolean filter;
(ii) for every $x \in L$, $x \vee x^* \in F$;
(iii) If $x^* \rightarrow x \in F$, then we have that $x \in F$.

5.4 The class $\mathcal{V} = \mathbf{H}$ (the subvariety of Heyting or $G-$ algebras)

Theorem 5.2 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:
(i) $F \in F(\mathbf{H})$;
(ii) F satisfies the condition (F_6);
(iii) F satisfies the condition (F_8);
(iv) If for every $x, y, z \in L$, $x \to (y \to z) \in F$, then we have $(x \to y) \to (x \to z) \in F$;
(v) for every $x, y \in L$, $(x \wedge (x \to y)) \to y \in F$;
(vi) for every $x, y \in L$, $(x \wedge y) \to (x \odot y) \in F$;
(vii) L/F is a Hilbert algebra;
($viii$) L/F is a Tarski algebra.

5.5 The class $\mathcal{V} = \mathbf{MV}$ (the subvariety of $MV-$ algebras)

Theorem 5.3 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:
(i) $F \in F(\mathbf{MV})$;
(ii) F satisfies the condition (F_9);
(iii) for every $x, y \in L$, $((x \to y) \to y) \to ((y \to x) \to x) \in F$.

5.6 The class $\mathcal{V} = \mathbf{MTL}$ (the subvariety of $MTL-$ algebras)

Theorem 5.4 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:
(i) $F \in F(\mathbf{MTL})$;
(ii) for every $x, y \in L$, $(x \to y) \vee (y \to x) \in F$;
(iii) If $x \to (y \vee z) \in F$, then we have that $(x \to y) \vee (x \to z) \in F$;
(iv) for every $x, y, z \in L$, $(x \to (y \vee z)) \to ((x \to y) \vee (x \to z)) \in F$;
(v) If $(y \wedge z) \to x \in F$, then we have $(y \to x) \vee (z \to x) \in F$;
(vi) for every $x, y, z \in L$, $((y \wedge z) \to x) \to ((y \to x) \vee (z \to x)) \in F$;
(vii) If $x \to z \in F$, then we have that $(x \to y) \vee (y \to z) \in F$;
($viii$) for every $x, y, z \in L$, $(x \to z) \to ((x \to y) \vee (y \to z)) \in F$;
(ix) If $(x \to y) \to z \in F$, then we have $((y \to x) \to z) \to z \in F$;
(x) for every $x, y, z \in L$, $((x \to y) \to z) \to (((y \to x) \to z) \to z) \in F$.

5.7 The class $\mathcal{V} = \mathbf{IRL}$ (the subvariety of involutive residuated lattices)

Theorem 5.5 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:

(i) $F \in F(\mathbf{IRL})$;

(ii) If for every $x, y \in L$, $x^* \to y^* \in F$, then we have $y \to x \in F$;

(iii) If for every $x, y \in L$, $x^* \to y \in F$, then we have $y^* \to x \in F$.

5.8 The class $\mathcal{V} = **-\mathbf{GA}$ (the subvariety of $**-G-$algebras)

Theorem 5.6 For $F \in \mathcal{F}_i(L)$, the following conditions are equivalent:

(i) $F \in F(**-\mathbf{GA})$;

(ii) If for every $x, y, z \in L$, $x^{**} \to (y \to z) \in F$, then $(x^{**} \to y) \to (x^{**} \to z) \in F$;

(iii) If for every $x, y \in L$, $x^{**} \to (x^{**} \to y) \in F$, then $x^{**} \to y \in F$;

(iv) for every $x \in L$, $x^{**} \to (x^{**})^2 \in F$.

5.9 The class $\mathcal{V} = \mathbf{SgF}$ (of semi-G-filters in a residuated lattice)

Proposition 5.5 For a i-filter F of L the following assertions are equivalent:

(i) F is a semi-G-filter;

(ii) for every $x \in L$, $(x \wedge x^*)^* \in F$;

(iii) If $x \to x^* \in F$, then e have that $x^* \in F$.

5.10 The class $\mathcal{V} = \mathbf{StF}$ (of Stonean filters in a residuated lattice)

Theorem 5.7 Let L be a residuated lattice. We consider the following assertions:

(i) L is a Stonean residuated lattice;

(ii) L is a semi-G-algebra.

Then $(i) \Rightarrow (ii)$.

If $L \in \mathbf{MTL}$, then $(i) \Leftrightarrow (ii)$.

5.11 The class $\mathcal{V} = \mathbf{MTLF}, \mathbf{DivF}$ and BLF (of MTL - filters, divisible filters and BL - filters in a residuated lattice)

Proposition 5.9 For a i-filter F of a residuated lattice L, the following conditions are equivalent:

(i) $F \in \mathbf{MTLF}(\mathbf{L})$;
(ii) If $x \to (y \vee z) \in F$, then we have $(x \to y) \vee (x \to z) \in F$;
(iii) for every $x, y, z \in L$, $(x \to (y \vee z)) \to ((x \to y) \vee (x \to z)) \in F$;
(iv) If $(y \wedge z) \to x \in F$, then we have $(y \to x) \vee (z \to x) \in F$;
(v) for every $x, y, z \in L$, $((y \wedge z) \to x) \to ((y \to x) \vee (z \to x)) \in F$;
(vi) If $x \to z \in F$, then we have $(x \to y) \vee (y \to z) \in F$;
(vii) for every $x, y, z \in L$, $(x \to z) \to ((x \to y) \vee (y \to z)) \in F$;
$(viii)$ If $(x \to y) \to z \in F$, then we have $((y \to x) \to z) \to z \in F$;
(ix) for every $x, y, z \in L$, $((x \to y) \to z) \to (((y \to x) \to z) \to z) \in F$.

Proposition 5.10 For a i-filter F of a residuated lattice L, the following conditions are equivalent:

(i) $F \in \mathbf{DivF}(\mathbf{L})$;
(ii) for every $x, y \in L$, $(x \wedge y) \to [x \odot (x \to y)] \in F$;
(iii) If $z \to (x \wedge y) \in F$, then we have $z \to [x \odot (x \to y)] \in F$;
(iv) If $(x \to y) \to (x \to z) \in F$, then we have that $(x \wedge y) \to z \in F$.

Proposition 5.12 For a i-filter F of a residuated lattice L the following conditions are equivalent:

(i) $F \in \mathbf{BLF}(\mathbf{L})$;
(ii) If $(x \to y) \to (x \to z) \in F$, then we have that $(x \to z) \vee (y \to z) \in F$;
(iii) for every $x, y, z \in L$, $((x \to y) \to (x \to z)) \to ((x \to z) \vee (y \to z)) \in F$.

5.12 The class $\mathcal{V} = \mathbf{MVF}$ and RF (of MV- filters and regular filters in a residuated lattice)

Lemma 5.2 If $F \in \mathbf{MVF}(\mathbf{L})$, then $x^{**} \to x \in F$, for every $x \in L$.

Proposition 5.14 For a filter F of a residuated lattice L the following conditions are equivalent:

(i) $F \in \mathbf{RF}(\mathbf{L})$;
(ii) for every $x \in L$, $x^{**} \to x \in F$;
(iii) for every $x, y \in L$, $(y^* \to x^*) \to (x \to y) \in F$;
(iv) for every $x, y \in L$, $(x^* \to y) \to (y^* \to x) \in F$.

5.13 Applications

In this section we present some particular results about filters in $\mathcal{RL}$.

Theorem 5.8 ([22]) Let $\mathcal{V}$ be a subvariety of $\mathcal{RL}$ and $F, G \in \mathcal{F}_i(L)$. Then
(i) If $F, G \in F(\mathcal{V})$, then we have $F \cap G \in F(\mathcal{V})$;
(ii) If $F \subseteq G$ and $F \in F(\mathcal{V})$, then we have $G \in F(\mathcal{V})$.

Conclusions and future work

In this section I would like to recommend a series of new important directions of research based on the results developed in this thesis.

Problem 6.1 Find necessary and sufficient conditions for distributvity in commutative, respectively non-commutative residuated lattices.

I propose the following

Problem 6.2 Find another characterization for strong elements in a bounded integral residuated lattice.

Problem 6.3 Find other study of normal residuated lattices following the papers Normal Lattices ([37]) by William H. Cornish and Characterizations of normal lattices ([105]) by Y. S. Pawar.

I propose the following

Problem 6.4 Find other classes of residuated lattices which verify that $\mathbf{StF}(\mathbf{L}) = \mathbf{SgF}(\mathbf{L})$.

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